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Gromov-Witten invariants and modular forms

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Abstract

This is a PRELIMINARY and UNFINISHED set of personal notes. They are not intended as an exhaustive presentation of any particular topic, and in particular they do not include a complete set of references.

Contents

1 Introduction

A good warm-up for the relation between GW invariants and modular forms is the example of the torus, the simplest nontrivial CY manifold. We want to compute the generating functionals of GW invariants of genus g in the class d

$$F_g = \sum_{g,d} N_{g,d} q^d \tag{1.1}$$

where $q = \mathrm{e}^{2\pi i t}$, and t is the complexified Kähler class of the torus. It is useful to put all of these invariants in a single generating function

$$Z(q, g_s) = \exp\Big(\sum_{g=1}^{\infty} g_s^{2g-2} F_g\Big). \tag{1.2}$$

The invariants vanish for $g = 0$.

This generating functional turns out to have a simple representation. This is due to the connection of this theory to the counting of branched coverings (or, in physics, to YM in two dimensions). In particular, it is known that this partition function can be computed as

$$\widehat{Z}(q, g_s) = \sum_{d=0}^{\infty} q^d \sum_{R,\, \ell(R)=d} e^{g_s \kappa_R}, \tag{1.3}$$

where $\widehat{Z}(q, g_s) = q^{\frac{1}{24}} Z(q, g_s)$, the second sum is over Young tableaux with d boxes, and κ_R is given by

$$\kappa_R = \sum_i l_i (l_i - 2i + 1), \tag{1.4}$$

where l_i are the lengths of rows in the Young tableau. The above formula has been rederived recently in [9]. By using the fermionic picture of $U(\infty)$ representation theory, this quantity can be computed immediately as

$$\widehat{Z}(q, g_s) = \oint \frac{dz}{2\pi i z} \prod_{n=1}^{\infty} (1 + z q^{n-1/2} e^{g_s (n-1/2)^2/2})(1 + z^{-1} q^{n-1/2} e^{-g_s (n-1/2)^2/2}), \tag{1.5}$$

see [11] for the details of this derivation. One consequence of this formula is that the F_g are deeply related to modular forms, where the q variable is the one introduced above. For example, if one uses the product formula

$$\vartheta_3(z|\tau) = \sum_{n \in \mathbf{Z}} q^{n^2/2} z^n = \prod_{n=1}^{\infty} (1 - q^n)(1 + z q^{n-1/2})(1 + z^{-1} q^{n-1/2}), \tag{1.6}$$

one finds that

$$F_1(q) = -\log \eta(q). \tag{1.7}$$

With some more work one finds

$$F_2(q) = \frac{1}{103680}(10E_2^3 - 6E_2E_4 - 4E_6), \tag{1.8}$$

and in general one has that F_g are quasimodular forms of weight $6g - 6$.

This is a very beautiful result and confirms many ideas about mirror symmetry in the very simple case of $\mathbf{T}^2$. Interestingly enough, the phenomenon persists in higher dimensions, although we are still far from writing compact formulae like (1.5) for the full generating functional. The physics behind this is based on heterotic/type II duality, and it was proposed already some time ago by Kachru and Vafa [18]. For this relation to modularity to hold, one has to consider a special class of CY threefolds, namely K3 fibrations [22].

2 K3 surfaces

A mandatory reference for K3 surfaces in string theory is the excellent review [2].

2.1 Some general results on algebraic surfaces

This subsection is mostly based on standard references on algebraic surfaces [5, 16, 4].

Algebraic surfaces have numerical invariants which are preserved under birational maps. Given an algebraic surface X, its *irregularity* is defined as

$$q(X) = h^1(X, \mathcal{O}_X) \tag{2.1}$$

and one has

$$2q(X) = b_1(X). \tag{2.2}$$

The *geometric genus* $p_g(X)$ is defined as

$$p_g(X) = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}(K_X)), \tag{2.3}$$

where K_X is the canonical line bundle of X and we have used Serre duality. For algebraic surfaces, we have the general relation

$$b_2^+(X) = 2p_g(X) + 1 \tag{2.4}$$

The *holomorphic Euler characteristic* is defined as

$$\chi(\mathcal{O}_X) = \sum_n (-1)^n h^n(X, \mathcal{O}_X), \tag{2.5}$$

and for an algebraic surface it is given by

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X). \tag{2.6}$$

Noether's formula relates the holomorphic Euler characteristic to the usual topological Euler characteristic $\chi(X)$

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi(X)). \tag{2.7}$$

An important object in Algebraic Geometry is the set of complex line bundles on a complex manifold M. It is called the *Picard group* or *Picard lattice* of M and it can be defined as

$$\mathrm{Pic}(M) = H^1(M, \mathcal{O}_M^*), \tag{2.8}$$

where $\mathcal{O}_M^*$ is the sheaf of nowhere vanishing holomorphic functions on M. The Picard group can be given another description by using the short exact sequence of sheaves

$$0 \to \mathbf{Z} \to \mathcal{O}_M \to \mathcal{O}_M^* \to 0, \tag{2.9}$$

which leads to the long sequence in sheaf cohomology

$$0 \to H^1(M, \mathbf{Z}) \to H^1(M, \mathcal{O}_M) \to \mathrm{Pic}(M) \to H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O}_M) \to \cdots \tag{2.10}$$

The map

$$c_1 : \mathrm{Pic}(M) \to H^2(M, \mathbf{Z}) \tag{2.11}$$

is given by taking the first Chern class of the line bundle. The kernel of this map is isomorphic to the image of $H^1(M, \mathcal{O}_M) \simeq H^{(0,1)}(M)$ in $\mathrm{Pic}(M)$. Another useful result is that the map

$$H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O}_M) \tag{2.12}$$

is obtained as ([16], p. 163)

$$H^2(M, \mathbf{Z}) \to H^2(M, \mathbf{C}) \to H^{(0,2)}(M), \tag{2.13}$$

where the last map is the projection of the de Rham cohomology of M with complex coefficients onto the $(0, 2)$ part.

Another way of thinking about $\mathrm{Pic}(M)$ is as an extension

$$0 \to T \to \mathrm{Pic}(M) \to NS(M) \to 0 \tag{2.14}$$

where

$$T = \frac{H^1(M, \mathcal{O}_M)}{H^1(M, \mathbf{Z})} \tag{2.15}$$

is a complex torus, and $NS(M) \subset H^2(M, \mathbf{Z})$ is the so-called Néron-Severi lattice of M (see [5], p. 6).

We will be interested in the particular case in which M is simply connected, therefore $H^{(0,1)}(M) = 0$, therefore $T = 0$, and (2.11) is injective. Since the kernel of (2.12) is $H^{(1,1)}(M)$, we find

$$\mathrm{Pic}(M) \simeq H^2(M, \mathbf{Z}) \cap H^{(1,1)}(M). \tag{2.16}$$

If we denote

$$\rho(M) = \mathrm{rk}\,\mathrm{Pic}(M), \tag{2.17}$$

we have that

$$\rho(M) \leq h^{1,1}(M). \tag{2.18}$$

The rank of the Picard lattice, also called the *Picard number* of M, depends on the complex structure.

If M is an algebraic manifold, all classes in $H^{(1,1)}(M)$ can be realized in terms of holomorphically embedded curves (this is the Lefschetz theorem on $(1,1)$-classes, see [16] pp. 162–4). Therefore, the Picard lattice can be also defined as

$$\mathrm{Pic}(M) = \{C \in H^2(M, \mathbf{Z}) : C \quad \text{holomorphically} \quad \text{embedded}\}. \tag{2.19}$$

For $M = \mathbb{P}^k$, one has $\rho(\mathbb{P}^k) = 1$ and it is generated by the hyperplane line bundle $[H]$. We will also denote

$$x = c_1([H]), \tag{2.20}$$

which is the hyperplane class.

Finally, we will need some general results on hypersurfaces in projective spaces. The total Chern class of $\mathbb{P}^k$ is given by

$$c(\mathbb{P}^k) = (1 + x)^{k+1}, \tag{2.21}$$

and the canonical line bundle to $\mathbb{P}^k$ is given by ([16], p. 146)

$$K_{\mathbb{P}^k} = [-(k+1)H]. \tag{2.22}$$

The *adjunction formula* says that the dual normal bundle $N^*_X V$ to a hypersurface (which is a line bundle) is given by

$$N^*_V = [-V]\big|_V, \tag{2.23}$$

where $[D]$ denotes the line bundle associated to a divisor. On the other hand, it is well-known (see [16], pp. 165–7) that, if V is given by the zero locus of a homogeneous polynomial of degree d, then

$$[V] = [dH]. \tag{2.24}$$

This is because homogeneous polynomials of degree d are in one-to-one correspondence with global sections of H^d:

$$\mathrm{Sym}^d(\mathbf{C}^{k+1*}) \simeq H^0(\mathbb{P}^k, \mathcal{O}(H^d)), \tag{2.25}$$

and this space has dimension

$$\binom{d+k}{k}. \tag{2.26}$$

Therefore, if the divisor $[V]$ is given by the zero locus of a homogeneous polynomial of degree d, it can be represented as the zero locus of a section of H^d, and (2.24) follows.

Another standard result we will use is the *Lefschetz hyperplane theorem*. We first recall a few standard facts. A line bundle L on a complex manifold is *positive* if there exists a metric on L such that its curvature is a positive $(1,1)$-form i.e. a form that leads to a positive-definite Hermitian inner product in the holomorphic tangent space. The basic example of a positive line bundle is the hyperplane bundle H on $\mathbb{P}^k$, since $\mathrm{i}x/(2\pi)$ in (2.20) is precisely the $(1,1)$ form associated to the Fubini-Study metric on $\mathbb{P}^k$ ([16], p. 150).

The Lefschetz hyperplane theorem goes as follows ([16], p. 156). Let $V \subset M$ be a smooth hypersurface of an n-dimensional compact, complex manifold M with $L = [V]$ positive. Then, the map

$$H^q(M, \mathbf{Q}) \to H^q(V, \mathbf{Q}) \tag{2.27}$$

induced by the inclusion $V \hookrightarrow M$, is an isomorphism for $q \geq n-2$ and injective for $q = n-1$. This theorem applies in particular to the case in which V is a hypersurface of the projective space $\mathbb{P}^k$ given by a homogeneous polynomial of degree $d > 0$. If $k > 2$, then it follows from the hyperplane theorem and the simple-connectedness of projective spaces that $b_1(V) = 0$. We conclude that *any hypersurface in projective space of complex dimension* ≥ 2 *is simply-connected.*

We will also need some results on algebraic curves on algebraic surfaces. Consider a curve C on a surface X. The adjunction formula states that

$$K_C = (K_X + C)\big|_C. \tag{2.28}$$

The genus of C is given by

$$g(C) = \frac{1}{2}(2 - \chi(C)), \tag{2.29}$$

and the Euler characteristic can be computed as minus the degree of the canonical bundle to C,

$$\chi(C) = -\deg K_C = -\int_C c_1(K_C) = -K_X \cdot C - C^2, \tag{2.30}$$

so we find the genus formula

$$g(C) = \frac{K_X \cdot C + C^2}{2} + 1. \tag{2.31}$$

Using Riemann-Roch one can also compute

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) + \frac{1}{2}(C^2 - C \cdot K_X) = g(C) + \hat{d} + \chi(\mathcal{O}_X) - 1, \tag{2.32}$$

where

$$\hat{d} = -K_X \cdot C \tag{2.33}$$

is the degree of the curve C with respect to the anticanonical class of X. The deformation space of the curve is given by $\mathbb{P}H^0(C)$, and if the appropriate vanishing holds, the rank of $H^0(C)$ can be computed from $\chi(\mathcal{O}_C)$. We then find that the deformation space for C is a projective space given by

$$\mathbb{P}^{\chi(\mathcal{O}_C)-1}. \tag{2.34}$$

Example. For $X = \mathbb{P}^2$, let us define the degree d of the curve as

$$[C] = d[H], \tag{2.35}$$

where H is the hyperplane class. This means that the homology class of C in $\mathbb{P}^2$ is given by dH. From (2.22) we have

$$K_X = -3H \longrightarrow g(C) = \frac{1}{2}(d-1)(d-2), \tag{2.36}$$

where we also used that $H^2 = 1$ (i.e. two lines intersect generically at one point). In this case,

$$\chi(\mathcal{O}_C) = \frac{1}{2}(d+1)(d+2). \tag{2.37}$$

2.2 Elementary properties of K3 surfaces

2.2.1 Definition and numerical invariants

In Algebraic Geometry, a K3 surface X is a compact, Kähler surface which has $q(X) = 0$ and trivial canonical bundle,

$$K_X = 0. \tag{2.38}$$

This information is enough to determine all numerical and topological invariants of X. First, by (2.2) we have that a K3 surface is simply-connected. From (2.38) we find

$$c_1(X) = 0. \tag{2.39}$$

Therefore, a K3 surface is a Calabi-Yau manifold of complex dimension 2. From this condition it also follows that

$$h^{2,0} = 1 \qquad \text{or} \qquad p_g(X) = 1. \tag{2.40}$$

With this we can already compute the holomorphic Euler characteristic of a K3 surface:

$$\chi(\mathcal{O}_X) = 2, \tag{2.41}$$

and from (2.4) we find that for a K3 surface

$$b_2^+(X) = 3. \tag{2.42}$$

Using now Noether's theorem (2.7) we find

$$\chi(X) = 24. \tag{2.43}$$

From here we immediately deduce

$$b_2(X) = 22, \qquad \sigma(X) = -16. \tag{2.44}$$

and we check

$$3\sigma(X) + 2\chi(X) = K_X^2 = 0. \tag{2.45}$$

We can now write the Hodge diamond of a K3 surface as

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{2.46}$$

As a lattice, the second cohomology of a K3 surface is of the form

$$H_2(X, \mathbf{Z}) = \Gamma^{3,19}, \tag{2.47}$$

where the inner product Q is simply the cup product in cohomology, and we will denote $Q(\alpha, \beta) = (\alpha, \beta)$. It follows from Poincaré duality that the two-cohomology lattice of a four-manifold is unimodular (*i.e.* $\det Q = \pm 1$). On the other hand, for four-manifolds one has that

$$(\alpha, \alpha) = (c_1(X), \alpha) \mod 2. \tag{2.48}$$

This follows from Wu's formula and the fact that $c_1(X)$ is an integer lift of $w_2(X)$ (see [14], Chapter 1). Since $c_1(X) = 0$ for a K3 surface, it follows that the lattice (2.47) is even. Standard results on the classification of unimodular even lattices (see for example [14], chapter 1) show that the lattice (2.47) is of the form

$$\Gamma^{3,19} = 2\,E_8(-1) \oplus 3\,\Gamma^{1,1}, \tag{2.49}$$

where $\Gamma^{1,1}$ is the unimodular even lattice with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.50}$$

2.2.2 K3 surface as a CY manifold

Since a K3 surface is a two-dimensional CY manifold, it has a globally defined, nowhere vanishing holomorphic $(2,0)$ form Ω. This form is in fact the generator of $H^{(2,0)}(X)$. As a form in $H^2(X,\mathbf{C})$, we can write it as

$$\Omega = x + \mathrm{i}y, \tag{2.51}$$

where x, y are in $H^2(X,\mathbb{R})$. From the obvious identities

$$\int_X \Omega \wedge \Omega = 0, \qquad \int_X \Omega \wedge \overline{\Omega} = \int_X |\Omega|^2 > 0 \tag{2.52}$$

one deduces

$$x \cdot y = 0, \qquad x^2 = y^2 > 0. \tag{2.53}$$

This means that x, y are linearly independent vectors which span a spacelike plane in $H^2(X,\mathbb{R})$. This plane will be also denoted by Ω. As in the case of CY threefolds, the periods of Ω parameterize the deformations of complex structure.

2.2.3 Lattices

The subspace $H^{2,+}(X,\mathbb{R})$ of self-dual forms on X is spanned by the Kähler form J and the form Ω. The fact that J is spacelike is due to the fact that

$$\mathrm{Vol}(X) = \int_X J \wedge J > 0. \tag{2.54}$$

This also follows from the decomposition

$$H^{2,+}(X,\mathbf{C}) = \mathbf{C} \cdot J \oplus H^{2,0}(X,\mathbf{C}) \oplus H^{0,2}(X,\mathbf{C}) \tag{2.55}$$

which is valid for two-dimensional Kähler manifolds. We will denote by Σ the space $H^{2,+}(X,\mathbb{R})$ regarded as a three-dimensional plane in $H^2(X,\mathbb{R})$.

It is convenient to divide the lattice $H^2(X, \mathbf{Z})$ into the Picard lattice and its complement Λ, called the *transcendental lattice*:

$$H^2(X, \mathbf{Z}) = \mathrm{Pic}(X) \oplus \Lambda, \tag{2.56}$$

where Λ has rank $22 - \rho(X)$. Notice that the maximal Picard number of a K3 surface is

$$\rho(X) \leq 20. \tag{2.57}$$

For an *algebraic* K3 surface, i.e. a K3 surface that can be embedded in a projective space through algebraic equations, the Picard number is at least one. This is because the Kähler form inherited from the projective space is an integer $(1,1)$ form. Since this is a spacelike element of $H^{2,+}(X, \mathbb{R})$, the number of spacelike elements in $\mathrm{Pic}(X)$ is at least one. It can not be greater than this, however, since we know that Σ is spanned by J and Ω, which is *not* in the Picard lattice (it has Dolbeault bidegree $(2,0)$). We then conclude that *the Picard lattice of any algebraic K3 surface has signature*

$$(1, \rho(X) - 1). \tag{2.58}$$

It then follows that the transcendental lattice Λ has signature

$$(2, 20 - \rho(X)), \tag{2.59}$$

where the two spacelike directions correspond to Ω, which can be viewed as

$$\Omega = \Sigma \cap (\Lambda \otimes \mathbb{R}). \tag{2.60}$$

We can represent the full interplay of lattices and planes in figure 1.

2.3 First model of a K3 surface: quartic in $\mathbb{P}^3$

An explicit construction of a K3 surface can be made as follows. Since we want a surface, we can look for a projective hypersurface X in $\mathbb{P}^3$ of the form

$$x_0^d + x_1^d + x_2^d + x_3^d = 0. \tag{2.61}$$

Let us show that, for an appropriate choice of d, this is indeed a K3 surface. By the Lefshetz hyperplane theorem, this hypersurface is simply-connected. The only thing to impose is then that $c_1(X) = 0$. We first notice that the tangent bundle of $\mathbb{P}^3$ along X satisfies

$$T\mathbb{P}^3\big|_X = T_X \oplus N_X \tag{2.62}$$

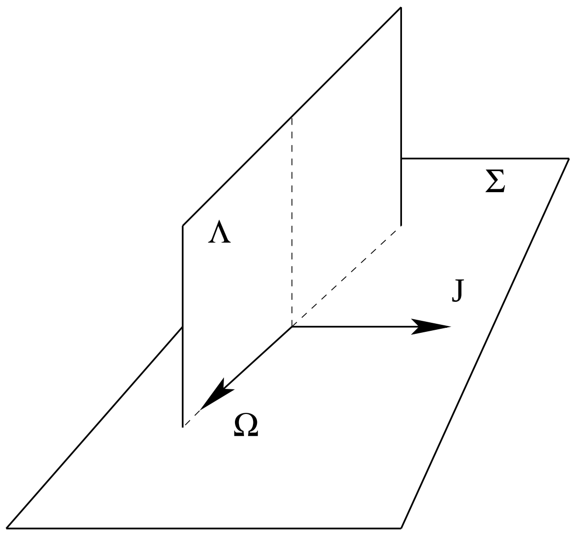

Figure 1: This represents the planes Σ and the plane spanned by the transcendental lattice Λ. The Kähler form is perpendicular to Λ and belongs to the Picard lattice. The intersection of Λ and Σ is the plane Ω.

where T_X and N_X are respectively the tangent and normal bundle to X. The total Chern class satisfies, by Whitney's formula,

$$c(\mathbb{P}^3) = c(T_X) \cdot c(N_X). \tag{2.63}$$

Using now (2.21), (2.24) and (2.23), we obtain for the hypersurface (2.61),

$$c(T_X) = \frac{(1+x)^4}{1+dx} = 1 + (4-d)x + \cdots . \tag{2.64}$$

Therefore, $c_1(X) = 0$ if and only if $d = 4$ in (2.61).

We can of course consider a more general quartic equation

$$\sum_{i+j+k+l=4} a_{ijkl} x_0^i x_1^j x_2^k x_3^l = 0. \tag{2.65}$$

This describes a family of K3 surfaces parameterized by $35 - 1 = 34$ parameters (the number 35 comes from (2.26) with $d = 4$, $k = 3$, and we subtract an overall rescaling of the equation).

2.4 Second model of K3: Kummer surface and orbifolds

Consider a torus $\mathbf{T}^2$, and the following $\mathbf{Z}_2$ action

$$z \to -z. \tag{2.66}$$

This has four fixed points,

$$(0,0),\ (0,1/2),\ (1/2,0),\ (1/2,1/2), \tag{2.67}$$

so the quotient $\mathbf{T}^2/\mathbf{Z}_2$ is an orbifold. Similarly, we can consider a four-dimensional orbifold by taking $\mathbf{T}^4 = \mathbf{T}_2 \times \mathbf{T}_2$ and the symmetry

$$(z_1, z_2) \to (-z_1, -z_z). \tag{2.68}$$

This has 16 fixed points, obtained by doing the Cartesian product of the two sets of fixed points (2.67). Let us consider the orbifold

$$X_{\rm o} = \mathbf{T}^4/\mathbf{Z}_2 \tag{2.69}$$

where the $\mathbf{Z}_2$ symmetry is given by (2.68). We have $b_1(X_{\rm o}) = 0$, since the basis of one forms of $\mathbf{T}^4$ given by dx_i, $i = 1, \cdots, 4$, is projected out by the $\mathbf{Z}_2$ symmetry. We have however six two-forms

$$dx_i \wedge dx_j \tag{2.70}$$

which are preserved by the quotient. This leads to

$$H^2(X_{\rm o}) = 3\Gamma^{1,1}. \tag{2.71}$$

This orbifold is clearly singular, since it has 16 singularities which are locally of the form

$$\mathbf{C}^2/\mathbf{Z}_2. \tag{2.72}$$

One can smooth out $X_{\rm o}$ by blowing up these singularities. This leads to 16 exceptional curves with self-intersection -2, and we end up with a K3 surface which is called a *Kummer surface.*

2.5 Moduli of K3 surfaces

We will now discuss various moduli spaces associated to K3 surfaces: the moduli space of complex structures, the moduli space of Einstein metrics, the moduli space of K3 surfaces with a B field, and the moduli spaces of algebraic K3 surfaces with a B field. Good references for this subsection are [2, 3, 27].

We start with the moduli space of complex structures. As in the CY case, this will be given (thanks to the Torelli theorem) by the possible periods of the homolorphic 2-form Ω. To compute the periods, we choose a basis $\{e_i\}_{i=1,\cdots,b_2(X)}$ of the lattice $H^2(X, \mathbf{Z})$. Such a choice gives a so-called *marked K3 surface*. The periods are given by

$$u^i = \int_{e_i} \Omega. \tag{2.73}$$

If we denote by $\{\omega_i\}_{i=1,\cdots,b_2(X)} \in H^2(X,\mathbf{Z})$ the dual 2-forms, such that

$$\int_{e_i} \omega_j = \delta^i_j, \tag{2.74}$$

then

$$\int_X \omega_i \wedge \omega_j = C_{ij}, \tag{2.75}$$

where C_{ij} is the intersection matrix of the lattice $H^2(X,\mathbf{Z})$. It follows from the above that we can write

$$\Omega = u^i \omega_i. \tag{2.76}$$

Notice that the $u^i \in \mathbf{C}^{3,19}$ are endowed with a natural metric induced by the lattice

$$\langle u, v\rangle = C_{ij} u^i v^j \tag{2.77}$$

with signature $(3,19)$. The period map can be regarded as a map

$$\pi : (X, e_i) \to \frac{O(3,19)}{O(2)\times O(1,19)} \tag{2.78}$$

where the coset space is parameterized by the u^i. To understand this in more detail, notice first that the form Ω is defined up to an overall rescaling by a nonzero scalar, so that the periods u^i are projective coordinates and are defined up to $u \sim \lambda u$ (this is as in the CY case). Also, due to (2.52), the periods u^i satisfy

$$\langle u, u\rangle = 0, \qquad \langle u, \overline{u}\rangle > 0, \tag{2.79}$$

and the image of the period map is then the set

$$\Big\{u \in \mathbf{C}^{3,19} : \langle u, u\rangle = 0, \quad \langle u, \overline{u}\rangle > 0\Big\}/(u \sim \lambda u). \tag{2.80}$$

But this is precisely a representation of the above coset. Another way to see this is that the period map gives the position of the spacelike 2-plane Ω inside $\mathbb{R}^{3,19}$, and this space is parameterized by the above Grassmannian. We then conclude

$$\boxed{\mathcal{M}_{\text{complex structures}}(\text{K3}) = \frac{O(3,19)}{O(2)\times O(1,19)}} \tag{2.81}$$

The moduli space of Einstein metrics can be obtained similarly thanks to the famous theorem of Yau. This theorem says that, if we fix the complex structure and the cohomology class of the Kähler form, there exists a unique Ricci-flat metric on X. But since the spacelike plane Σ is spanned by precisely these elements, a choice of Σ, the spacelike 3-plane in $\mathbb{R}^{3,19}$, will lead to a metric. The space of such planes is again a Grassmannian, and we deduce

$$\boxed{\mathcal{M}_{\text{Einstein metrics}}(\text{K3}) = \frac{O(3,19)}{O(3)\times O(19)}} \tag{2.82}$$

In string theory, it is important to include a B field to complexify the Kähler form as follows:

$$\omega = iJ + B, \tag{2.83}$$

with the condition that B is a $(1,1)$ form, and that J leads to a positive volume (i.e. that J is inside the Kähler cone). We now want to determine the moduli space of K3 surfaces together with a B field. We first define another period vector by considering the *full cohomology lattice* of the K3 surface:

$$H^*(X,\mathbf{Z}) = U \oplus H^2(X,\mathbf{Z}) = \Gamma^{4,20}, \tag{2.84}$$

where

$$U = H^0(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z}) \tag{2.85}$$

is a lattice of type $\Gamma^{1,1}$ lattice. We now associate to ω the generalized period

$$\omega \to \Pi_\omega = \big(1, -\frac{1}{2}\int_X \omega\wedge\omega, \int_{e_i}\omega, \big) \in \mathbf{C}^{4,20}. \tag{2.86}$$

Notice that

$$\text{Re}(\Pi_\omega) = (1, \frac{1}{2}(J\wedge J - B\wedge B), B), \qquad \text{Im}(\Pi_\omega) = (0, J\wedge B, J). \tag{2.87}$$

Notice that the real and imaginary parts of Π_ω, together with the real and imaginary parts of the standard period vector

$$\Pi_\Omega = \big(0, 0, \int_{e_i}\Omega\big) \tag{2.88}$$

span a four-dimensional spacelike plane Π in $\mathbb{R}^{4,20}$, endowed with the metric $\langle\,,\,\rangle$ inherited from the lattice $\Gamma^{4,20}$. This is easy to see by computing

$$\langle \Pi_\omega, \Pi_\omega\rangle = 0, \qquad \langle \Pi_\omega, \overline{\Pi}_\omega\rangle > 0, \tag{2.89}$$

or, equivalently,

$$\begin{aligned} &\langle \text{Re}(\Pi_\omega), \text{Re}(\Pi_\omega)\rangle = \langle \text{Im}(\Pi_\omega), \text{Im}(\Pi_\omega)\rangle = \int_X J\wedge J > 0, \\ &\langle \text{Re}(\Pi_\omega), \text{Im}(\Pi_\omega)\rangle = 0. \end{aligned} \tag{2.90}$$

These four vectors are independent, due to the second equality in (2.90) and

$$\langle \Pi_\Omega, \Pi_\omega \rangle = \int_X \Omega \wedge \omega = 0, \qquad \langle \overline{\Pi}_\Omega, \Pi_\omega \rangle = \int_X \overline{\Omega} \wedge \omega = 0, \tag{2.91}$$

which vanish by bidegree reasons: Ω and $\overline{\Omega}$ are $(2,0)$ and $(0,2)$ forms, respectively, while ω is a $(1,1)$ form. The moduli space of K3 surfaces with a B field will be now given by the moduli of spacelike planes Π in $\mathbb{R}^{4,20}$, which is again a Grassmannian. We conclude,

$$\boxed{\mathcal{M}_{\text{moduli and B field}}(\text{K3}) = \frac{O(4,20)}{O(4)\times O(20)}} \tag{2.92}$$

We finally consider the moduli space of *algebraic* K3 surfaces with a B field, which is the moduli space we will be mostly more interested about. On an algebraic surface, some of the lattice directions in $\Gamma^{3,19}$ belong to the Picard lattice, and they are of type $(1,1)$, therefore they cannot be paired with the holomorphic $(2,0)$ form Ω. This means that only the transcendental lattice can be paired to Ω, and only the Picard lattice can be paired to ω. To analyze the resulting moduli spaces, let us write

$$H^*(X, \mathbf{Z}) = \Upsilon \oplus \Lambda, \tag{2.93}$$

where

$$\Upsilon = U \oplus \text{Pic}(X) \tag{2.94}$$

is usually called the *quantum Picard lattice*. The quantum Picard lattice has signature $(2, \rho(X))$. We now choose bases for both the Picard lattice and the transcendental lattice, which will be denoted by $\{p_i\}_{i=1,\cdots,\rho(X)}$ and $\{t_i\}_{i=1,\cdots,22-\rho(X)}$. We consider now the periods

$$T_\Omega = \int_{t_i} \Omega \in \mathbf{C}^{2,20-\rho(X)}, \tag{2.95}$$

and

$$P_\omega = \big(1, -\frac{1}{2}\int_X \omega\wedge\omega, \int_{p_i}\omega, \big) \in \mathbf{C}^{2,\rho(X)}. \tag{2.96}$$

By the same arguments we have used before, we have that

$$\begin{aligned} \langle T_\Omega, T_\Omega \rangle &= 0, \qquad \langle T_\Omega, \overline{T}_\Omega \rangle > 0, \\ \langle P_\omega, P_\omega \rangle &= 0, \qquad \langle P_\omega, \overline{P}_\omega \rangle > 0 \end{aligned} \tag{2.97}$$

It follows that the periods (2.95) and (2.96) live in the corresponding cosets, and we conclude:

$$\boxed{\mathcal{M}_{\text{complex structures}}(\text{algebraic K3}) = \frac{O(2,20-\rho(X))}{O(2)\times O(20-\rho(X))}} \tag{2.98}$$

and

$$\boxed{\mathcal{M}_{\text{complexified Kahler}}(\text{algebraic K3}) = \frac{O(2,\rho(X))}{O(2)\times O(\rho(X))}} \tag{2.99}$$

2.6 Curves on K3 surfaces

Let C be a curve on a K3 surface. Since the canonical bundle of K3 is trivial, one finds the simple genus formula

$$g(C) = \frac{[C]^2}{2} - 1. \tag{2.100}$$

In particular, for a curve to have genus 0 (i.e. to be rational) we necessarily have

$$[C]^2 = -2. \tag{2.101}$$

These curves are very important since they correspond to simple roots in the cohomology lattice. They are responsible for enhanced gauge symmetries for string theory compactified on a K3 surface.

3 K3 fibrations

Some CY manifolds have the structure of a K3 fibration over $\mathbb{P}^1$, i.e. we have a map

$$\pi : M \to \mathbb{P}^1, \tag{3.1}$$

where M is the CY, and $\pi^{-1}(p) = X$, where X is a K3 surface. The Hodge number $h^{1,1}(M)$ of a CY with this structure is given by

$$h^{1,1}(M) = 1 + \rho^{\text{inv}}(X), \tag{3.2}$$

where the extra 1 comes from the Kähler class of the $\mathbb{P}^1$ base, and $\rho^{\text{inv}}(X)$ denotes the elements of the Picard lattice that are invariant when transported around the base. We will call these elements the monodromy invariant Picard lattice of X. There can be some extra contributions to (3.2) coming from "bad fibres", but we won't need that.

Consider now the compactification of type IIA theory on M. We will have $h^{1,1}(M)$ vector multiplets, which we will denote by

$$S, \qquad t^a, a = 1, \cdots, \rho^{\text{inv}}(X), \tag{3.3}$$

where S corresponds to the Kähler class of the $\mathbb{P}^1$ base, and t^a correspond to the monodromy invariant Picard lattice of the fibre. Let us denote by ω_S, ω_a the corresponding

$(1,1)$-cohomology classes. From the above geometry we also deduce that the only non-trivial triple intersection number on the CY is

$$\int_M \omega_S \wedge \omega_a \wedge \omega_b = \eta_{ab}, \tag{3.4}$$

where η_{ab} is the inner product on the monodromy invariant Picard lattice. Recall that the Picard lattice had signature (2.58), where the $+1$ direction corresponded to the Kähler form of the manifold. This form is monodromy invariant, since it is the restriction to the fiber of the Kähler form of M, and we conclude that η_{ab} has signature $(1, \rho^{\rm inv}(X) - 1)$. We conclude that the cubic term in the perturbative prepotential of the type IIA theory on M is given by

$$F_{\rm tree}(S, t^a) = S \eta_{ab} t^a t^b. \tag{3.5}$$

Of course, this prepotential receives worldsheet instanton corrections. At tree level, however, we see that the moduli space factorizes into the moduli space for the Kähler modulus of the base, and the moduli space of complexified Kähler forms in the fiber:

$$\mathcal{M}_{\rm IIA} = \mathcal{H} \times \frac{O(2, \rho^{\rm inv}(X))}{O(2) \times O(\rho^{\rm inv}(X))}, \tag{3.6}$$

where $\mathcal{H}$ is the upper half-plane, which parameterizes the moduli space for S, and we have used the result (2.99).

Example 1. A favorite example is the hypersurface $X_{24}(1,1,2,8,12)^{-480}$ in weighted projective space, described by the polynomial

$$x_1^2 + x_2^3 + x_3^{12} + x_4^{24} + x_5^{25} = 0. \tag{3.7}$$

This CY has the structure of a K3 fibration over $\mathbb{P}^1$. The base of the fibration is parameterized by the coordinates (x_4, x_5). By fixing x_4/x_5 and redefining $x_4 \to x_4^2$, one obtains the K3 fiber as a hypersurface in weighted $\mathbb{P}^3(1,1,4,6)$:

$$x_1^2 + x_2^3 + x_3^{12} + x_4^{12} = 0. \tag{3.8}$$

The elementary topological data of this CY are the following: we have $\chi(M) = -480$. The Picard lattice of the $K3$ fiber is of the form

$$\mathrm{Pic}(X) = \Gamma^{1,1} \tag{3.9}$$

therefore

$$h^{1,1}(M) = 3, \tag{3.10}$$

where the extra generator corresponds to the Kähler class of the base.

Example 2. Our second example is the CY

$$Y = (\mathrm{K3} \times \mathbf{T}^2)/\mathbf{Z}_2 \tag{3.11}$$

where the $\mathbf{Z}_2$ acts as the free Enriques involution [4] on the K3 and as the inversion $\mathbf{Z}_2 : z \to -z$ on the coordinate z of the two torus that we considered before, with four fixed points p_i. The geometry of the $\mathbf{T}^2/\mathbf{Z}_2$ orbifold is that of an $\mathbb{P}^1$ with four conical curvature singularities at the p_i each of which has deficit angle π. The total space M is a $K3$ fibration over the $\mathbb{P}^1$, and by construction it has Enriques fibres E over the four p_i. We will be particularly interested on the action of the Enriques involution on the two-cohomology lattice of the K3

$$H^2(\mathrm{K3}, \mathbf{Z}) = \Gamma^{9,1} \oplus \Gamma^{9,1} \oplus \Gamma^{1,1}_g. \tag{3.12}$$

This symmetry exchanges the first two factors, and acts as $p \to -p$ on $\Gamma^{1,1}_g$. The invariant part of this lattice is precisely the two-cohomology lattice of the Enriques surface (with doubled inner product due to the exchange symmetry), and it becomes the Picard lattice of the K3 fiber,

$$\mathrm{Pic}(X) = \Gamma^{1,1}(2) \oplus E_8(-2), \tag{3.13}$$

The Hodge numbers of Y can now be easily obtained by looking at the cohomology classes of the original manifold $\mathrm{K3} \times \mathbf{T}^2$ which are invariant under $\mathbf{Z}_2$. One finds,

$$h^{1,1}(Y) = 11, \qquad \chi(Y) = 0. \tag{3.14}$$

4 Heterotic theories

We will consider here heterotic string compactifications which lead to $\mathcal{N} = 2$ supersymmetry in four dimensions, obtained typically as orbifolds of toroidal compactifications. We briefly recall that heteroric strings are obtained by tensoring 26 free chiral bosons for the right-moving sector, while on the left moving sector we will consider 10 free chiral bosons together with their supersymmetric partners ψ. A toroidal compactification of the heterotic string down to four dimensions is obtained by considering the $(22, 6)$ compact directions to live in a lattice of the form

$$\Gamma^{22,6}. \tag{4.1}$$

4.1 The STU model

The STU model is a $\mathbf{Z}_2$ orbifold of the above toroidal compactification. We start with the following splitting of the lattice (4.1)

$$\Gamma^{22,6} \longrightarrow \Gamma^{2,2} \oplus \Gamma^{4,4} \oplus \Gamma^{0,8} \oplus \Gamma^{0,8}, \tag{4.2}$$

where the lattice $\Gamma^{0,8}$ is just $E_8(-1)$. The interpretation of this splitting is standard: $\Gamma^{2,2}$ and $\Gamma^{4,4}$ are lattices corresponding to toroidal compactifications on $\mathbf{T}^2$ and $\mathbf{T}^4$, respectively, and $E_8 \oplus E_8$ is the lattice corresponding to the internal gauge group. In order to understand the action on the E_8 lattice, it is useful to remember that this lattice, at the point of maximal gauge symmetry, can be represented by 8 complex fermions ψ^a, $a = 1, \cdots, 8$, along compact directions. This leads for example to the following formula for the theta function of the E_8 lattice:

$$\Theta_{E_8}(q) = \frac{1}{2} \sum_{a,b=0,1} \vartheta^8 [{}^a_b]. \tag{4.3}$$

The $\mathbf{Z}_2$ orbifold action acts on $\Gamma^{4,4}$ and on one of the E_8s. On $\Gamma^{4,4}$ it acts as the $\mathbf{Z}_2$ symmetry that we used to construct Kummer surfaces starting from $\mathbf{T}^4$

$$X^I, \psi^I \longrightarrow -X^I, -\psi^I, \qquad I = 1, 2, 3, 4 \quad \text{on } \Gamma^{4,4} \tag{4.4}$$

while on the E_8 it acts as a $\mathbf{Z}_2$ symmetry on two of the complex fermions:

$$\psi^{1,2} \rightarrow -\psi^{1,2}, \qquad \psi^a \rightarrow \psi^a, \quad a = 3, \cdots, 8 \quad \text{on } E_8. \tag{4.5}$$

This orbifold has a simple space-time interpretation: we have compactified the six internal directions of the heterotic string on a manifold of the form K3$\times\mathbf{T}^2$. The K3 factor comes from the quotient of $\Gamma^{4,4}$ by $\mathbf{T}_2$, and it is a singular K3 surface at the orbifold point of moduli space (of course, this is not a problem at all for the string propagation, as it has been known for a long time). The $\mathbf{T}^2$ factor comes from the $\Gamma^{2,2}$ lattice. The orbifold action on the E_8 breaks the gauge symmetry as

$$E_8 \times E_8 \longrightarrow E_8 \times E_7 \times SU(2), \tag{4.6}$$

and the $SU(2)$ which appears here is identified through the orbifold action with the $SU(2)$ of the holonomy of the K3 surface. This is the well-known embedding of the spin connection in the gauge connection, typical of heterotic compactifications on manifolds of special holonomy.

To check that this heterotic compactification is consistent, one has to verify modular invariance at one-loop. The one-loop partition function can be computed by using standard orbifold techniques (see [20], section 12.4, for more details):

1) bosons in the lattices $\Gamma^{2,2}$ and $\Gamma^{0,8}$, which are invariant under $\mathbf{Z}_2$, give the contribution

$$\frac{\Theta_{\Gamma^{2,2}}}{\eta^2 \bar{\eta}^2} \times \frac{\Theta_{\Gamma^{0,8}}}{\eta^8}. \tag{4.7}$$

2) Bosons in the lattice $\Gamma^{4,4}$ give different orbifold blocks $Z_{\Gamma^{4,4}}\big[{h \atop g}\big]$ with explicit expressions

$$Z_{\Gamma^{4,4}}\big[{0 \atop 0}\big] = \frac{\Theta_{\Gamma^{4,4}}}{\eta^4\bar\eta^4}, \qquad Z_{\Gamma^{4,4}}\big[{h \atop g}\big] = 2^4 \frac{\eta^2\bar\eta^2}{\vartheta^2\big[{1-h \atop 1-g}\big]\bar\vartheta^2\big[{1-h \atop 1-g}\big]} \tag{4.8}$$

3) The $\Gamma^{0,8}$ lattice where the symmetry acts leads to orbifold blocks $Z_{\Gamma^{0,8}}\big[{h \atop g}\big]$ which can be obtained by simply noticing that in the fermionized version of the $\Gamma^{0,8}$ lattice we are simply considering a $\mathbf{Z}_2$ orbifold along two complex directions. We then find

$$Z_{\Gamma^{0,8}}\big[{h \atop g}\big] = \frac{1}{2\eta^8} \sum_{\gamma,\delta=0,1} \vartheta\big[{\gamma+h \atop \delta+g}\big]\vartheta\big[{\gamma-h \atop \delta-g}\big]\vartheta^6\big[{\gamma \atop \delta}\big]. \tag{4.9}$$

which has the following explicit values for the different orbifold blocks:

$$\begin{aligned}
Z_{\Gamma^{0,8}}\big[{0 \atop 0}\big] =& \frac{1}{2\eta^8}(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = \frac{E_4(\tau)}{\eta^8}, \\
Z_{\Gamma^{0,8}}\big[{0 \atop 1}\big] =& \frac{1}{2\eta^8}\vartheta_3^2\vartheta_4^2(\vartheta_3^4 + \vartheta_4^4), \\
Z_{\Gamma^{0,8}}\big[{1 \atop 0}\big] =& \frac{1}{2\eta^8}\vartheta_3^2\vartheta_2^2(\vartheta_3^4 + \vartheta_2^4), \\
Z_{\Gamma^{0,8}}\big[{1 \atop 1}\big] =& \frac{1}{2\eta^8}\vartheta_2^2\vartheta_4^2(\vartheta_2^4 - \vartheta_4^4).
\end{aligned} \tag{4.10}$$

4) In the left-moving sector, we have fermions in the two transverse, uncompactified directions, together with fermions on the $\Gamma^{2,2}$ lattice. Each of these pairs leads to a factor $\vartheta\big[{a \atop b}\big]/\eta$. Finally, we have the fermions in the $\Gamma^{4,4}$ lattice. We then find

$$\frac{1}{2\bar\eta^4} \sum_{a,b=0,1} (-1)^{a+b+ab}\bar\vartheta^2\big[{a \atop b}\big]\bar\vartheta\big[{a+h \atop b+g}\big]\bar\vartheta\big[{a-h \atop b-g}\big] \tag{4.11}$$

5) Finally, the bosons in the transverse directions lead to a factor

$$\frac{1}{\tau_2\eta^2\bar\eta^2} \tag{4.12}$$

Putting everything together, we find the following partition function:

$$\begin{aligned}
Z =& \frac{1}{2}\sum_{h,g=0}^{2} \frac{\Theta_{\Gamma^{2,2}}\Theta_{\Gamma^{0,8}} Z_{\Gamma^{4,4}}\big[{h \atop g}\big]}{\tau_2\eta^{12}\bar\eta^4} \frac{1}{2\eta^8} \sum_{\gamma,\delta=0,1} \vartheta\big[{\gamma+h \atop \delta+g}\big]\vartheta\big[{\gamma-h \atop \delta-g}\big]\vartheta^6\big[{\gamma \atop \delta}\big] \\
&\times \frac{1}{2\bar\eta^4} \sum_{a,b=0,1} (-1)^{a+b+ab}\bar\vartheta^2\big[{a \atop b}\big]\bar\vartheta\big[{a+h \atop b+g}\big]\bar\vartheta\big[{a-h \atop b-g}\big].
\end{aligned} \tag{4.13}$$

It is an interesting exercise to check that this partition function is modular invariant, so that the compactification we have constructed is a consistent vacuum of the heterotic string. It has $\mathcal{N} = 2$ supersymmetry in four dimensions. This is easily seen since K3$\times\mathbf{T}^2$ has two covariantly constant spinors (it has holonomy $SU(2)$), twice as much as a generic CY of $SU(3)$ holonomy. The heterotic string has 16 supercharges in 10d, which decompose under compactification down to four dimensions as $\mathbf{4} \times (2, 2)$, where $\mathbf{4}$ is a chiral spinor in 6d,and the $(2, 2)$ are two Weyl spinors in 4d. For the holonomy $SU(2)$, only two out of the $\mathbf{4}$ are covariantly constant, therefore we produce two copies of the $\mathcal{N} = 1$ supersymmetry charge in four dimensions, i.e. we have $\mathcal{N} = 2$ SUSY.

The massless spectrum will organize itself into representations of the $\mathcal{N} = 2$ superalgebra. It is easy to see that we will have four $U(1)$ gauge fields from the $\mathbf{T}^2$ compactification, together with the gauge fields of the unbroken gauge group (4.6). One of the $U(1)$ gauge fields from the $\mathbf{T}^2$ compactification enters into the SUGRA multiplet as the graviphoton, while the other enter into vector multiplets. One of these vector multiplets is somewhat special since it contains as a complex scalar S the dilaton-axion fields:

$$S = \phi + ia, \tag{4.14}$$

where the axion a is obtained by dualizing the B field that lives in four dimensions. There are two other vector multiplets coming from the $\mathbf{T}^2$. Their complex scalars parameterize the moduli space of the Narain moduli space

$$\frac{O(2,2)}{O(2) \times O(2)} \simeq \mathcal{H} \times \mathcal{H}, \tag{4.15}$$

and they are denoted by T, U. We will be mostly interested in studying the dependence of the F_g amplitudes on these moduli.

4.2 The FHSV model

The heterotic FHSV model is an asymmetric orbifold of the heterotic string [13]. One first considers the splitting of the compactification lattice $\Gamma^{22,6}$ as

$$\Gamma_u = \Gamma^{9,1} \oplus \Gamma^{9,1} \oplus \Gamma_s^{1,1} \oplus \Gamma^{2,2} \oplus \Gamma_g^{1,1}, \tag{4.16}$$

where $\Gamma^{9,1}$ can be decomposed as

$$\Gamma^{9,1} = \Gamma_d^{1,1} \oplus E_8(-1). \tag{4.17}$$

We now act with a $\mathbf{Z}_2$ symmetry as follows:

$$|p_1, p_2, p_3, p_4, p_5\rangle \to e^{\pi i \delta \cdot p_3} |p_2, p_1, p_3, -p_4, -p_5\rangle \tag{4.18}$$

J	1	2	3
ζ_J	2	1/2	1/2
α_J	δ	0	δ
β_J	0	δ	δ

Table 1: ζ_J, α_J and β_J for the different blocks

where $\delta = (1,1) \in \Gamma_s^{1,1}$, and $\delta^2 = 2$. Therefore $\mathbf{Z}_2$ acts as an exchange symmetry in the direct sum $\Gamma^{9,1} \oplus \Gamma^{9,1}$, as a shift in $\Gamma_s^{1,1}$, and as -1 in $\Gamma^{2,2} \oplus \Gamma_g^{1,1}$. It is easy to see [13] that this asymmetric orbifold leads to an heterotic string compactification with $\mathcal{N} = 2$ supersymmetry in four dimensions. The massless spectrum consists of 11 vector multiplets, 11 hypermultiplets, and the supergravity multiplet.

The vector multiplet moduli space for this compactification is given by

$$\mathrm{SL}(2, \mathbf{Z}) \backslash \mathrm{SL}(2, \mathbb{R})/SO(2) \times \mathcal{M}, \tag{4.19}$$

where

$$\mathcal{M} = O(\Gamma_1) \backslash O(10,2)/[O(10) \times O(2)], \tag{4.20}$$

and $O(\Gamma_1)$ is the group of automorphisms of the lattice

$$\Gamma_1 = \Gamma_s^{1,1} \oplus \Gamma_d^{1,1}(2) \oplus E_8(-2). \tag{4.21}$$

This is in fact the lattice associated to the untwisted, projected sector of the orbifold.

We will now compute the one-loop partition function of the FHSV orbifold, since the results will be useful for the computation of the F_g amplitudes (this, as well as the helicity supertrace generating function, have been computed independently in [10]). We will denote by $Z\left[{h \atop g}\right]$ the partition functions on the sector twisted by h and with the g element inserted. Here, $g, h = 0, 1$ in the usual way.

Let us first consider the bosonic sector. In the untwisted, unprojected sector we simply have

$$Z^b\left[{0 \atop 0}\right] = \frac{1}{2\bar{\eta}^{24}(\tau)\eta^8(\tau)} \overline{\Theta}_{\Gamma^u}(\tau).$$

In order to consider the other sectors, we introduce the lattices Γ_J with $J = 1, 2, 3$:

$$\Gamma_J = \Gamma_s^{1,1} \oplus \Gamma_d^{1,1}(\zeta_J) \oplus E_8(-\zeta_J), \tag{4.22}$$

The values of ζ_J, α_J, β_J are given in table 4.2. The three different cases $J = 1, 2, 3$ correspond respectively to the orbifold blocks 01, 10 and 11. In the untwisted, projected sector we identify the two sets of bosonic excitations associated to the two $\Gamma^{9,1}$ lattices. This amounts to a doubling of the τ parameter in the nonzero modes [24]. We then find,

$$Z^b[{}^0_1] = \frac{4}{\bar\eta^9(2\tau)\eta(2\tau)\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{}^1_0](\tau)}\right|^3 \overline{\Theta}_{\Gamma_1}(\tau,\delta,0).$$

For the 10 and 11 orbifold blocks we find

$$\begin{aligned} Z^b[{}^1_0] =& \frac{4}{\bar\eta^9(\tau/2)\eta(\tau/2)\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{}^0_1](\tau)}\right|^3 \overline{\Theta}_{\Gamma_2}(\tau,0,\delta), \\ Z^b[{}^1_1] =& \frac{4}{\bar\eta^9(\frac{\tau+1}{2})\eta(\frac{\tau+1}{2})\bar\eta^3(\tau)\eta^3(\tau)}\left|\frac{\eta(\tau)}{\vartheta[{}^0_0](\tau)}\right|^3 \overline{\Theta}'_{\Gamma_3}(\tau,\delta,\delta). \end{aligned}$$

In the 11 block, the $'$ in the theta function indicates that the sum over lattice vectors includes an insertion of

$$(-1)^{v^2}, \tag{4.23}$$

where v is the projection of p onto $\Gamma^{1,1}(\frac{1}{2}) \oplus E_8(-\frac{1}{2})$.

Let us now consider the fermionic sector in detail. The fermions in the $\Gamma^{1,1}_s$ lattice do not change under the $\mathbf{Z}_2$ symmetry, so together with the fermions in the uncompactified directions we have

$$Z^f_{\Gamma^{1,1}_s}[{}^a_b] = \left(\frac{\vartheta[{}^a_b](\tau)}{\eta(\tau)}\right)^{3/2}.$$

The orbifold blocks for two complex fermions with symmetry $\psi \to -\psi$ are given by (see for example [20], eq. (12.4.15)):

$$\frac{\vartheta[{}^{a+h}_{b+g}](\tau)\vartheta[{}^{a-h}_{b-g}](\tau)}{\eta^2}$$

Therefore, for the fermions in $\Gamma^{2,2}\oplus\Gamma^{1,1}_g$ one finds

$$Z^f_{\Gamma^{2,2}\oplus\Gamma^{1,1}_g}[{}^h_g][{}^a_b] = \left(\frac{\vartheta[{}^{a+h}_{b+g}](\tau)\vartheta[{}^{a-h}_{b-g}](\tau)}{\eta^2}\right)^{3/4}.$$

The treatment of the two fermions coming from $\Gamma^{9,1}\oplus\Gamma^{9,1}$ is slightly more delicate. The 00 block in the a,b sector is simply

$$Z^f_{\Gamma^{9,1}\oplus\Gamma^{9,1}}[{}^0_0][{}^a_b] = \frac{\vartheta[{}^a_b](\tau)}{\eta(\tau)}.$$

Let us now analyze the invariant states in the NS sector. A convenient basis for the Hilbert space $\mathcal{H}_{\rm NS}^{(1)} \otimes \mathcal{H}_{\rm NS}^{(2)}$ is given by

$$\left(\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k}}\psi^{(2)}_{-m_1}\cdots\psi^{(2)}_{-m_l} \pm (1\leftrightarrow 2)\right)|0\rangle,$$
$$\left(\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k+1}}\psi^{(2)}_{-m_1}\cdots\psi^{(2)}_{-m_{2l+1}} \mp (1\leftrightarrow 2)\right)|0\rangle,$$

where $n_i, m_i > 0$ are half-integers. The above states have the sign ±1, respectively, under the $\mathbf{Z}_2$ symmetry generator g which exchanges the two lattices. It is easy to see that in computing the trace over the Hilbert space with an insertion of g, the above states cancel except when the (1) and the (2) content is the same. Therefore, only the states

$$\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k+1}}\psi^{(2)}_{-n_1}\cdots\psi^{(2)}_{-n_{2k+1}}|0\rangle,$$
$$\psi^{(1)}_{-n_1}\cdots\psi^{(1)}_{-n_{2k}}\psi^{(2)}_{-n_1}\cdots\psi^{(2)}_{-n_{2k}}|0\rangle$$

contribute to the trace, with signs -1 and $+1$ under g, respectively. An odd number of fermion oscillators leads to a -1 sign, but this is like having an insertion of $(-1)^F$. We then have

$$\mathrm{Tr}_{\mathcal{H}_{\rm NS}^{(1)}\otimes\mathcal{H}_{\rm NS}^{(2)}}\, g\, q^{L_0-c/24} = \mathrm{Tr}_{\mathcal{H}_{NS}}(-1)^F\, q^{2L_0-c/12} = \left(\frac{\vartheta[{0\atop 1}](2\tau)}{\eta(2\tau)}\right)^{\frac12},$$

where the doubling in τ is due to the doubling in the oscillator content. Notice that the insertion of $(-1)^F$ in the above trace does not change anything, since $(-1)^{F_1}$ and $(-1)^{F_2}$ cancel each other. We then find:

$$Z^f_{\Gamma^{9,1}\oplus\Gamma^{9,1}}[{0\atop 1}][{a\atop b}] = \left(\frac{\vartheta[{a\atop 1}](2\tau)}{\eta(2\tau)}\right)^{\frac12},$$

and the expressions for the other blocks can be obtained by modular transformations.

Putting all these results together, we can write up the one-loop partition functions for the different blocks. One finds, for example:

$$Z[{0\atop 0}] = \frac{1}{2\bar\eta^{24}(\tau)\eta^8(\tau)}\overline{\Theta}_{\Gamma_u}(\tau)\sum_{a,b}(-1)^{a+b+ab}\left(\frac{\vartheta\left[{a\atop b}\right](\tau)}{\eta(\tau)}\right)^4.$$

for the 00 block. For the 01 block, one finds

$$\begin{aligned} Z[{0\atop 1}] =& \frac{4}{\bar\eta^9(2\tau)\eta(2\tau)|\eta(\tau)|^3}\frac{1}{|\vartheta\left[{1\atop 0}\right](\tau)|^3}\overline{\Theta}_{\Gamma_1}(\tau,\delta,0) \\ &\times\frac{\left(\vartheta[{0\atop 0}](\tau)\right)^{3/2}\left(\vartheta[{0\atop 1}](\tau)\right)^{3/2}}{\eta^3(\tau)}\frac{\left(\vartheta[{0\atop 0}](2\tau)\right)^{1/2}-\left(\vartheta[{0\atop 1}](2\tau)\right)^{1/2}}{\left(\eta(2\tau)\right)^{\frac12}}. \end{aligned} \tag{4.24}$$

5 Computing F_g amplitudes in heterotic string theory

5.1 Computing the integrand

The general expression for the F_g couplings in these compactifications is given by the one-loop integral [1]

$$F_g = \int_{\mathcal{F}} d^2\tau \tau_2^{2g-1} \frac{1}{|\eta|^4} \sum_{\text{even}} \frac{i}{\pi} \partial_\tau \left(\frac{\vartheta[^a_b](\tau)}{\eta(\tau)} \right) Z_g^{\text{int}}[^a_b]. \tag{5.1}$$

In this equation, the integration is over the fundamental domain of the torus,

$$Z_g^{\text{int}}[^a_b] = \langle : \left(\partial X\right)^{2g} : \rangle \tag{5.2}$$

is a correlation function evaluated in the internal conformal field theory, and X is the complex boson corresponding to the right-moving modes on the $\mathbf{T}^2$. The evaluation of the correlation function reduces to zero modes [1], and the final result involves insertions of the right-moving momentum p_R. For this reason, it is convenient to introduce the Narain theta function with an insertion,

$$\Theta_\Gamma^g(\tau, \alpha, \beta) = \sum_{p \in \Gamma} p_R^{2g-2} \exp\left\{ \pi i \tau (p + \beta/2)_+^2 + \pi i \overline{\tau} (p + \beta/2)_-^2 + \pi i (p + \beta/2, \alpha) \right\}. \tag{5.3}$$

In general the internal CFT will be an orbifold theory and we will have to consider different orbifold blocks, which will be labeled by J. For each of these blocks there is a different Narain lattice Γ_J with different α_J, β_J, and we will denote

$$\Theta_J^g = \Theta_{\Gamma_J}^g(\tau, \alpha_J, \beta_J). \tag{5.4}$$

The integral (5.1) can now be written as

$$F_g = \int_{\mathcal{F}} d^2\tau \tau_2^{2g-1} \sum_J \mathcal{I}_J^g, \tag{5.5}$$

where

$$\mathcal{I}_J^g = \frac{\mathcal{P}_g(q)}{Y^{g-1}} \overline{\Theta}_J^g(\tau) f_J(q). \tag{5.6}$$

In this equation, $\mathcal{P}_g(q)$ is defined by [1, 25]

$$\left(\frac{2\pi\eta^3\lambda}{\vartheta_1(\lambda|\tau)} \right)^2 = \sum_{g=0}^{\infty} (2\pi\lambda)^{2g} \mathcal{P}_g(q), \tag{5.7}$$

and $f_J(q)$ is a modular form which depends on the details of the internal CFT. As usual, we write $q = \exp(2\pi i\tau)$. Finally, the quantity Y in (5.6) is a moduli-dependent function related to the Kähler potential as $K = -\log Y$. The quantities $\mathcal{P}_g(q)$ can be explicitly written in terms of generalized Eisenstein series. To do this, one uses the expansion

$$\frac{2\pi\eta^3 z}{\vartheta_1(z|\tau)} = -\exp\left[\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}E_{2k}(\tau)z^{2k}\right]. \tag{5.8}$$

If we now introduce the polynomials $\mathcal{S}_k$ through:

$$\exp\Big[\sum_{n=1}^{\infty}x_n z^n\Big] = \sum_{n=0}^{\infty}\mathcal{S}_n(x_1,\ldots,x_n)z^n, \tag{5.9}$$

we can easily check that $\mathcal{P}_g(q)$ is a modular form of weight $(2g,0)$ given by

$$\mathcal{P}_g(q) = \mathcal{S}_g\bigg(x_k = \frac{|B_{2k}|}{(2k)!}E_{2k}(q)\bigg). \tag{5.10}$$

where B_{2k} are Bernoulli numbers, and $E_{2k}(q)$ is the Eisenstein series introduced in (A.35). We have, for instance,

$$\mathcal{P}_1(q) = \frac{1}{12}E_2(q), \quad \mathcal{P}_2(q) = \frac{1}{1440}(5E_2^2 + E_4). \tag{5.11}$$

Example 1. *The STU model.* We now give some details about the integrand of F_g and the functions $f_J(q)$ for the STU model that we analyzed in the previous section. We first compute the fermionic contribution

$$\frac{i}{\pi\bar\eta^3}\sum_{\text{even}}(-1)^{a+b+ab}\overline{\partial}_{\bar\tau}\left(\frac{\bar\vartheta[^a_b](\bar\tau)}{\bar\eta(\bar\tau)}\right)\bar\vartheta^2[^a_b]\bar\vartheta[^{a+h}_{b+g}]\bar\vartheta[^{a-h}_{b-g}] \tag{5.12}$$

for the different orbifold blocks. We find the following results:

$$\begin{aligned}
&h = g = 0: && 0,\\
&h = 1,\, g = 0: && \frac{1}{4\bar\eta^8}\bar\vartheta_3^2\bar\vartheta_2^2\bar\vartheta_4^4,\\
&h = 0,\, g = 1: && -\frac{1}{4\bar\eta^8}\bar\vartheta_3^2\bar\vartheta_4^2\bar\vartheta_2^4,\\
&h = 1,\, g = 1: && \frac{1}{4\bar\eta^8}\bar\vartheta_4^2\bar\vartheta_2^2\bar\vartheta_3^4.
\end{aligned} \tag{5.13}$$

We then see that the untwisted, unprojected sector of the orbifold does not contribute. Putting this together with the other ingredients (namely, bosons in $\Gamma^{4,4}$ and the contribution from the $\Gamma^{0,8}$s) one finds that the modular form $f(q)$ is given by

$$\frac{4E_4}{\eta^{18}}\left\{\frac{\vartheta_2^2\vartheta_4^2}{\vartheta_3^2}(\vartheta_2^4 - \vartheta_4^4) - \frac{\vartheta_3^2\vartheta_4^2}{\vartheta_2^2}(\vartheta_3^4 + \vartheta_4^4) + \frac{\vartheta_2^2\vartheta_3^2}{\vartheta_4^2}(\vartheta_2^4 + \vartheta_3^4)\right\} = -\frac{2E_4E_6}{\eta^{24}}. \tag{5.14}$$

Example 2. *The FHSV model.* This is rather straightforward using the results of the previous section. We have four orbifold blocks, but the first block (corresponding to $h = g = 0$) vanishes. The blocks $(h, g) = (0, 1), (1, 0), (1, 1)$ will be labeled by $J = 1, 2, 3$, and an easy computation shows that the modular forms $f_J(q)$ in (5.6) are given by

$$\begin{aligned} f_1(q) &= -\frac{128}{\eta^6(\tau)\vartheta_2^6(\tau)} = \frac{2}{q}\prod_{n=1}^{\infty}(1-q^{2n})^{-12}, \\ f_2(q) &= \frac{4}{\eta^6(\tau)\vartheta_4^6(\tau)} = 4q^{-\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^n)^{-12}(1-q^{n-1/2})^{-12}, \\ f_3(q) &= \frac{4}{\eta^6(\tau)\vartheta_3^6(\tau)} = 4q^{-\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^n)^{-12}(1+q^{n-1/2})^{-12}. \end{aligned} \tag{5.15}$$

The Narain lattices for $J = 1, 2, 3$ are given in (4.22), and the corresponding theta functions in (5.6) are the same ones that appear in the computation of the one-loop partition function in the previous section.

5.2 Computing the integral

In order to complete the evaluation of F_g, one has to perform the one-loop integral (5.1). This step is more involved and requires the techniques introduced in [12, 17] in the context of string threshold corrections, further refined in [8, 26], and used to compute the F_g couplings in [25]. The basic idea is to perform a *lattice reduction.* We will consider for simplicity the case in which the integral involves only one Narain lattice Γ, with $\alpha = \beta = 0$. Let z be a primitive vector of Γ of zero norm, and let $K = (\Gamma \cap z^\perp)/\mathbf{Z}z$. This lattice, which has signature $(b^+ - 1, b^- - 1)$, is called the reduced lattice. A typical situation when choosing a reduction vector occurs when the lattice Γ has $\Gamma^{1,1}$ as a sublattice. In this case, one can take z to be one of the vectors that generate $\Gamma^{1,1}$. In the reduced lattice one can construct "reduced" projections $\widetilde{P}$ as follows: consider $z_\pm \equiv P_\pm(z)$, and decompose $\mathbb{R}^{b^\pm} \simeq \langle z_\pm \rangle \oplus \langle z_\pm \rangle^\perp$. The projection on the orthogonal complement $\langle z_\pm \rangle^\perp$ is the reduced projection $\widetilde{P}_\pm$. It can be explicitly written in terms of $P_\pm$ as

$$\widetilde{P}_\pm(\lambda) = P_\pm(\lambda) - \frac{(P_\pm(\lambda), z_\pm)}{z_\pm^2} z_\pm. \tag{5.16}$$

The computation of the integral (5.5) through lattice reduction involves writing the vectors of the lattice Γ as

$$p = nz + mz' + p^K, \tag{5.17}$$

where p^K is a vector in the reduced lattice K. When the reduction vector belongs to a sublattice $\Gamma^{1,1}(l)$, the vector z' is the other generator of the sublattice. After a Poisson resummation over n, the integral involves a sum over two integers, ℓ and m. Using modular invariance one can set $m = 0$. The result of these manipulations is a complicated expression for (5.5) which can be found in [25]. Howevee, the holomorphic limit

$$\bar{t} \to \infty, \qquad t \text{ fixed}, \tag{5.18}$$

leads to a rather simple expression for F_g. We define the coefficients $c_g(n)$ through

$$\mathcal{P}_g(q) f(q) = \sum_n c_g(n) q^n. \tag{5.19}$$

We then have [25]

$$F_g(t) = \sum_{r>0} c_g(r^2/2) \sum_{\ell=1}^{\infty} \ell^{2g-3} \mathrm{e}^{-\ell r \cdot y}. \tag{5.20}$$

In this equation, r^2 is computed with the norm of the reduced lattice K, and the restriction $r > 0$ means that we only consider vectors such that $\mathrm{Im}(r \cdot y) > 0$. Notice that the sum over ℓ in (5.20) can be written as

$$\mathrm{Li}_{3-2g}(\mathrm{e}^{-\nu^{-1} r \cdot y}), \tag{5.21}$$

where Li_n is the polylogarithm of index n defined as

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \tag{5.22}$$

The answer for the integral (5.1) obtained with the method of lattice reduction is only valid in the region of moduli space where $|z_+| \ll 1$. Different choices of regions in moduli space lead to different choices of lattice reduction, and therefore to different expressions for the topological ampltiudes. Notice that the expression for F_g in the heterotic computations involves a model-dependent quantity (the modular form $f(q)$) as well as the universal factors $\mathcal{P}_g$.

The computation of the one-loop integral is more subtle when one has arbitrary α, β shifts in the Narain theta lattice. It is possible however to generalize the lattice reduction technique to these cases. An example of this is the FHSV model, which is considered in detail in [23].

6 Enumerative geometry and modular forms

6.1 Gopakumar-Vafa invariants

As Gopakumar and Vafa discovered [15], the topological string amplitudes F_g have an underlying structure in terms of BPS invariants associated to D0-D2 bound states. If we consider the generating functional

$$F(\lambda) = \sum_{g=0}^{\infty} F_g(t)\lambda^{2g-2}, \tag{6.1}$$

then one has the following formula for the worldsheet instanton part of F_g:

$$F(\lambda) = \sum_{g=0}^{\infty}\sum_{d=1}^{\infty}\sum_{r\in H_2(M,\mathbf{Z})} n_g(r)\frac{1}{d}\left(2\sin\frac{d\lambda}{2}\right)^{2g-2} \mathrm{e}^{-dr\cdot t}, \tag{6.2}$$

where $n_g(r)$ are the Gopakumar-Vafa (GV) invariants which count bound states of D2-D0 branes. Notice that the sum over d in (6.2) plays the same role as the sum over ℓ in the heterotic computation (5.20). In the computation of Gopakumar and Vafa, this sum is the Poisson resummation of a sum over D0 brane charges. When $\nu = 1$, the product representation of $\vartheta_1(\nu|\tau)$ and the Gopakumar-Vafa representation (6.2) enable us to express the heterotic prediction of (5.20) as a formula for the generating functional of GV invariants:

$$\sum_{r\in \mathrm{Pic}(\mathrm{K3})} n_g(r) z^g q^{r^2/2} = f(q)\xi^2(z), \tag{6.3}$$

where Pic(K3) is the cohomology of the K3 fiber, $z = 4\sin^2(\lambda/2)$, and $\xi(z)$ is the function that appears in helicity supertraces (see for example [20])

$$\xi(z) = \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{1-2q^n\cos\lambda+q^{2n}} = \prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^n)^2+zq^n}. \tag{6.4}$$

GV invariants can be given a geometric interpretation in certain cases in terms of Euler characteristics of moduli spaces of *embedded* curves [15, 19]. The main formula derived in [19] is the following. Let $\mathcal{M}_{\delta,Q}$ be the moduli space of curves of genus g and with δ nodes in the two-homology class Q. Then,

$$n_Q^{g-\delta} = (-1)^{\dim\mathcal{M}_{\delta,Q}}\chi(\mathcal{M}_{\delta,Q}). \tag{6.5}$$

Alternatively, one has the following formula

$$n_Q^{g-\delta} = (-1)^{(\dim(\mathcal{M}_C)+\delta)}\sum_{p=0}^{\delta} b_{g-p,\delta-p}\chi(\mathcal{C}^{(p)}), \quad b_{g,k} := \frac{2}{k!}\prod_{i=1}^{k-1}(2g-(k+2)+i), \quad b_{g,0} := 1\ . \tag{6.6}$$

Here, $\mathcal{C}^{(p)}$ is the moduli space of the curve C in the class Q together with a choice of p points, which correspond to nodes of C. In particular $\mathcal{C}^{(0)} = \mathcal{M}_C$, and for curves of maximal genus g in the class Q one has

$$n_Q^g = (-1)^{\dim\mathcal{M}_C}\chi(\mathcal{M}_C). \tag{6.7}$$

When the curve is embedded in an algebraic surface X, the maximal genus is related to the degree of the curve through the formula (2.31).

The expression (6.6) reduces the computation of GV invariants to the problem to calculating the Euler numbers $\chi(\mathcal{C}^{(\delta)})$. When C is embedded in an algebraic surface X these can be computed as follows. If we force the smooth curve C to pass through δ given points in X, corresponding to the locations of the nodes, we impose δ linear constraints on its moduli space $\mathcal{M} = \mathbb{P}^{\chi(\mathcal{O}_C)-1}$. The moduli space of deformations is therefore reduced to $\mathbb{P}^{\chi(\mathcal{O}_C)-\delta-1}$. On the other hand we are free to choose the position of the points, which are therefore part of the moduli space of the nodal curves. The moduli space for a choice of n-points on X is naively X^n. Since the points are undistinguishable one considers the orbifold $\mathrm{Sym}^n(X) = X^n/S_n$ by the symmetric group S_n . The relevant model for the moduli space of n points is the "free field" resolution $\mathcal{X}_n = \mathrm{Hilb}^n(X)$ of this orbifold. The name comes from the fact that the Euler numbers of the resolved spaces are generated by a free field representation

$$\sum_{n=0}^{\infty} \chi(\mathcal{X}_n) q^n = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n}\right)^{\chi(X)} . \tag{6.8}$$

This is special bosonic case of a formula of Göttsche and Soergel [14], which gives the Poincaré polynomial of $\mathcal{X}_n$ in terms of bosonic and fermionic free fields. This means that we have a fibration

$$\begin{array}{ccc} \mathbb{P}^{\chi(\mathcal{O}_C)-\delta-1} & \longrightarrow & \mathcal{C}^{(\delta)} \\ & & \downarrow \\ & & \mathcal{X}_\delta \end{array} \tag{6.9}$$

Since this fibration is trivial one finds

$$\chi(\mathcal{C}^{(\delta)}) = (\chi(\mathcal{O}_C) - \delta)\chi(\mathcal{X}_\delta) . \tag{6.10}$$

From this formula one can in principle derive many of the relevant GV invariants.

6.2 GV invariants for the STU model

According to [18], the heterotic STU model is dual to type IIA theory on the CY hypersurface $X_{24}(1,1,2,8,12)^{-480}$. Therefore, the heterotic F_g should encode information about the GW invariants of these manifold. We want to check now some of the heterotic predictions by using the formulae of the previous section for GV invariants. We first present the heterotic prediction of [25], and then give the geometric interpretation of [21].

In the STU model, it was shown in [25] that the formula (5.20) applies with the modular form $f(q)$ given by (5.14). It is then simple to extract the GV invariant from this expression, and one finds

g	$r=0$	1	2	3	4	5	6	7	8
0	−2	480	282888	17058560	477516780	8606976768	115311621680	1242058447872	11292809553810
1	0	4	−948	−568640	−35818260	−1059654720	−20219488840	−286327464192	−3251739174540
2	0	0	−6	1408	856254	55723296	1718262980	34256077056	506823427338
3	0	0	0	8	−1860	−1145712	−76777780	−2455936800	−50899848132
4	0	0	0	0	−10	2304	1436990	98987232	3276127128
5	0	0	0	0	0	12	−2740	−1730064	−122357100
6	0	0	0	0	0	0	−14	3168	2024910
7	0	0	0	0	0	0	0	16	−3588
8	0	0	0	0	0	0	0	0	−18

We consider curves in the K3 fiber. We choose the Kähler parameters to be parameterized by

$$t_1 = U, \qquad t_2 = T - U. \tag{6.11}$$

We will denote by $[E]$ and $[B]$ the corresponding homology classes, which will have intersection form

$$[E]^2 = 0, \quad [E][B] = 1, \quad [B]^2 = -2. \tag{6.12}$$

This is easily derived by taking into account that the original basis associated to T, U has intersection form $\Gamma^{1,1}$. Therefore, by the genus formula for K3 surfaces (2.100), we see that $[E]$ is an elliptic curve while $[B]$ is a two-sphere. These two curves provide in fact a structure of elliptic fibration for the K3 fiber itself:

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{K3} \\ & & \downarrow \\ & & B \end{array} \tag{6.13}$$

Writing now

$$[C] = d_1[E] + d_2[B], \tag{6.14}$$

we find

$$[C]^2 = d_1 d_2 - d_2^2 + 1. \tag{6.15}$$

Let us now consider curves in the class $(d_1, d_2) = (g, 1)$, which will have by the above arithmetic genus g. What is the moduli space for such curves? The moduli for "motions" inside the K3 fiber is given by (2.34), which gives in this case

$$\chi(\mathcal{O}_C) = g + 1, \tag{6.16}$$

and we used that for a K3 surface $\chi_h(X) = 2$. But there is an extra transverse space for motion, given by the $\mathbb{P}^1$ in the base of the K3 fibration of the CY. We then find,

$$\mathcal{M}_C = \mathbb{P}^g \times \mathbb{P}^1. \tag{6.17}$$

The formula (6.6) gives then

$$n^{g}_{(g,1,0)} = (-1)^{g+1} 2(g+1). \tag{6.18}$$

This fits perfectly with the heterotic prediction.

We can now look at nodal curves, with effective genus $g-1$. This is now slightly different from what we discussed above. The space of curves with one marked point $\mathcal{C}^{(1)} = \mathcal{C}$ is given by a fibration over the full CY M:

$$\begin{array}{ccc} \mathbb{P}^{g-1} & \longrightarrow & \mathcal{C} \\ & & \downarrow \\ & & M \end{array} \tag{6.19}$$

and one finds

$$\chi(\mathcal{C}) = \chi(\mathbb{P}^{g-1})\chi(M) = -480g. \tag{6.20}$$

This leads to

$$n^{g-1}_{(g,1,0)} = (-1)^{g+1}(2(2g-2)(g+1) - 480g), \tag{6.21}$$

which also fits the heterotic prediction.

6.3 GV invariants for the FHSV model

According to [13], the heterotic FHSV model we described above should be dual to type IIA theory on the CY obtained as a $\mathbf{Z}_2$ quotient of K3 $\times$ $\mathbf{T}^2$

The FHSV model is much more subtle for various reasons. The final formula for F_g has been computed in [23] and reads

$$F_g(t) = \sum_{r>0} c_g(r^2) \left\{ 2^{3-2g} \mathrm{Li}_{3-2g}(e^{-r\cdot t}) - \mathrm{Li}_{3-2g}(e^{-2r\cdot t}) \right\}, \tag{6.22}$$

where

$$\sum_n c_g(n) q^n = f_1(q) \mathcal{P}_g(q) \tag{6.23}$$

and $f_1(q)$ is given in (5.15).

Due to the peculiar form in which the polylog function appears, the results for the GV invariants depend on the parity of the entries in r. If at least one entry in $r \in \Gamma$ is odd the second term in (6.22) does not contribute and we get the invariants $n_g^{odd}(r)$ listed in the table below. Note that $r^2 \in 2\mathbf{Z}$ because Γ is even.

If all entries in r are even then $r^2 \in 8\mathbf{Z}$ and we call the class r even. In the (6.22) the second terms gives a subleading correction to $n_g^{even}(r)$. The first few $n_g^{even}(r)$ are listed in Table 6.3.

g	$r^2=0$	2	4	6	8	10	12	14	16
0	0	0	0	0	0	0	0	0	0
1	8	128	1152	7680	42112	200448	855552	3345408	12166272
2	0	-16	-288	-2880	-21056	-125280	-641664	-2927232	-12166272
3	0	0	24	480	5264	41760	267360	1463616	7096992
4	0	0	0	-32	-704	-8400	-71872	-492800	-2872512
5	0	0	0	0	40	960	12384	113728	831960
6	0	0	0	0	0	-48	-1248	-17312	-169920
7	0	0	0	0	0	0	56	1568	23280
8	0	0	0	0	0	0	0	-64	-1920
9	0	0	0	0	0	0	0	0	72

Table 2: BPS invariants $n_g^{odd}(r)$ for the odd classes r in the fiber direction.

g	$r^2=0$	8	16	24	32
0	0	0	0	0	0
1	4	42048	12165696	1242726144	69636018752
2	0	-21024	-12165696	-1864089216	-139272037504
3	0	5256	7096656	1708748448	174090046880
4	0	-704	-2872416	-1158884992	-165915421248
5	0	40	831948	611668944	127601309256
6	0	0	-169920	-254819136	-80867605120
7	0	0	23280	83673040	42545564896
8	0	0	-1920	-21406464	-18592299200
9	0	0	72	4174920	6721882484
10	0	0	0	-598848	-1994908928
11	0	0	0	59472	480175264
12	0	0	0	-3648	-92117568
13	0	0	0	104	13732280
14	0	0	0	0	-1531072

Table 3: BPS invariants for the even classes r in the fiber direction.

We now use the geometric interpretation of the GV invariants to test some of the heterotic predictions of the FHSV model. The moduli space $\mathcal{M}_C$ factorizes for these curves into $\mathcal{M}_C(F)$ parametrizing movements of C in the fibre and $\mathbb{P}^1$ parametrizing movements of C over the base of Y. The $\mathbb{P}^1$ is therefore a component of the moduli space. Since away from the fixed points the topology of the base is $\mathbf{T}^2$, we will only have contributions from curves in the Enriques fiber, whose GV invariants $n_g(r)$ will include an overall factor $\chi(\mathbb{P}^1) = 2$.

It is therefore sufficient to consider curves in the four special Enriques fibres to explain the BPS invariants. We will denote the Enriques surface by E. Let us first recall an important fact about curves in an Enriques surfaces from [4]. According to proposition 16.1, for every such C in the class r in the Kähler cone there is a second curve $C + K_E$ in the class r up to torsion with $|C + K_E| \neq \emptyset$ and $r^2 = [C]^2 = [C + K_E]^2$. So each curve in the Enriques fibre is effectively doubled. Since we have four fibers we expect that the numbers in table 2 are divisible by eight, which is indeed the case. Let us now compute the moduli space of deformations $\mathcal{M}_C$ for smooth curves of genus g. Since $\chi_h(E) = 1$, we now have

$$\mathcal{M}_C = \mathbb{P}^{g-1} \ . \tag{6.24}$$

We apply now (6.6) and get for smooth curves in the class r of genus $g = \frac{r^2}{2} + 1$ in the class r

$$n_g(r) = 8 \cdot (-1)^{\frac{r^2}{2}} \chi(\mathbb{P}^{\frac{r^2}{2}}) = 8 \cdot (-1)^{\frac{r^2}{2}} (\frac{r^2}{2} + 1) \tag{6.25}$$

in agreement with table 2.

Let us now consider nodal curves. The generating function for Euler characteristics of the Hilbert scheme of points on the Enriques surface is given by

$$\sum_{n=0}^{\infty} \chi(\mathcal{M}_n) q^n = \prod_{n=1}^{\infty} \left(\frac{1}{1 - q^n} \right)^{\chi(E)} = 1 + 12q + 90q^2 + 520q^3 + 2535q^4 + \dots \tag{6.26}$$

and if we insert this result in (6.10) we reproduce immediately, and to a large extent, the heterotic predictions in table 2. The deviations between the two calculations are given in table 4.

These deviations are typical of the geometric approach of [19] and correspond to situations involving reducible curves and other topological complications.

g	$r^2=0$	2	4	6	8	10	12	14	16
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	24	288	2160	12544	61608
2	0	0	0	0	0	0	0	-32	-384
3	0	0	0	0	0	0	0	0	0

Table 4: Differences between the heterotic BPS prediction in table 2 and the geometric BPS calculation using (6.6) and (6.10).

A Theta functions and modular forms

Our conventions for the Jacobi theta functions are:

$$\begin{aligned}
\vartheta_1(\nu|\tau) &= \vartheta[^1_1](\nu|\tau) = i\sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}(n+1/2)^2}e^{i\pi(2n+1)\nu},\\
\vartheta_2(\nu|\tau) &= \vartheta[^1_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}(n+1/2)^2}e^{i\pi(2n+1)\nu},\\
\vartheta_3(\nu|\tau) &= \vartheta[^0_0](\nu|\tau) = \sum_{n\in\mathbf{Z}} q^{\frac{1}{2}n^2}e^{i\pi 2n\nu},\\
\vartheta_4(\nu|\tau) &= \vartheta[^0_1](\nu|\tau) = \sum_{n\in\mathbf{Z}}(-1)^n q^{\frac{1}{2}n^2}e^{i\pi 2n\nu},
\end{aligned} \tag{A.27}$$

where $q = e^{2\pi i\tau}$. When $\nu = 0$ we will simply denote $\vartheta_2(\tau) = \vartheta_2(0|\tau)$ (notice that $\vartheta_1(0|\tau) = 0$). The theta functions $\vartheta_2(\tau)$, $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$ have the following product representation:

$$\begin{aligned}
\vartheta_2(\tau) &= 2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2,\\
\vartheta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2,\\
\vartheta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2
\end{aligned} \tag{A.28}$$

and under modular transformations they behave as:

$$\begin{aligned}
\vartheta_2(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), & \vartheta_2(\tau+1) &= \mathrm{e}^{i\pi/4}\vartheta_2(\tau),\\
\vartheta_3(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), & \vartheta_3(\tau+1) &= \vartheta_4(\tau),\\
\vartheta_4(-1/\tau) &= \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), & \vartheta_4(\tau+1) &= \vartheta_3(\tau).
\end{aligned} \tag{A.29}$$

The theta function $\vartheta_1(\nu|\tau)$ has the product representation

$$\vartheta_1(\nu|\tau) = -2q^{\frac{1}{8}} \sin(\pi\nu) \prod_{n=1}^{\infty} (1-q^n)(1-2\cos(2\pi\nu)q^n + q^{2n}). \tag{A.30}$$

We also have the following useful identities:

$$\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau), \tag{A.31}$$

and

$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\,\eta^3(\tau), \tag{A.32}$$

where

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{A.33}$$

is the Dedekind eta function. One has the following doubling formulae

$$\begin{aligned} \eta(2\tau) =& \sqrt{\frac{\eta(\tau)\vartheta_2(\tau)}{2}}, \qquad \vartheta_2(2\tau) = \sqrt{\frac{\vartheta_3^2(\tau) - \vartheta_4^2(\tau)}{2}}, \\ \vartheta_3(2\tau) =& \sqrt{\frac{\vartheta_3^2(\tau) + \vartheta_4^2(\tau)}{2}}, \qquad \vartheta_4(2\tau) = \sqrt{\vartheta_3(\tau)\vartheta_4(\tau)}. \end{aligned} \tag{A.34}$$

The generalized Eisenstein series are defined by

$$E_{2n}(\tau) = 1 - \frac{4n}{B_{2n}} \sum_{k=1}^{\infty} \frac{k^{n-1}q^{2k}}{1-q^{2k}}, \tag{A.35}$$

where B_m are the Bernoulli numbers. The doubling formula for $E_2(\tau)$ is

$$E_2(2\tau) = \frac{1}{2}E_2(\tau) + \frac{1}{4}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau)). \tag{A.36}$$

Under modular transformations, modular forms transform as follows:

$$S:\ \mathrm{Im}\,\tau \to \frac{\mathrm{Im}\,\tau}{|\tau|^2}. \tag{A.37}$$

Under an S transformation, Jacobi theta functions transform as

$$\begin{aligned} \vartheta_2(-1/\tau) =& \sqrt{-i\tau}\vartheta_4(\tau) \\ \vartheta_3(-1/\tau) =& \sqrt{-i\tau}\vartheta_3(\tau) \\ \vartheta_4(-1/\tau) =& \sqrt{-i\tau}\vartheta_2(\tau) \end{aligned} \tag{A.38}$$

and under T as

$$\begin{aligned}\vartheta_2(\tau+1) =& e^{i\pi/4}\vartheta_2(\tau)\\ \vartheta_3(\tau+1) =& \vartheta_4(\tau)\\ \vartheta_4(\tau+1) =& \vartheta_3(\tau).\end{aligned} \tag{A.39}$$

The η function transforms as

$$\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau). \tag{A.40}$$

B CFT at one-loop

Here we gather some useful formula for doing one-loop computations in CFT. Narain theta functions are defined by

$$\Theta_\Gamma(\tau,\alpha,\beta) = \sum_{p\in\Gamma} \exp\left\{\pi i\tau(p+\beta/2)_+^2 + \pi i\overline{\tau}(p+\beta/2)_-^2 + \pi i(p+\beta/2,\alpha)\right\} \tag{B.41}$$

When $\alpha=\beta=0$, we will simply write $\Theta_\Gamma(\tau)$. The one-loop partition function for bosons compactified on a lattice Γ of signature (b^+,b^-) is given by

$$\frac{\Theta_\Gamma(\tau)}{\eta^{b^+}\bar{\eta}^{b^-}} \tag{B.42}$$

Fermions with boundary conditions (a,b) have partition function

$$Z\left[{a \atop b}\right] = \sqrt{\frac{\vartheta\left[{a \atop b}\right]}{\eta}}, \qquad a,b=0,1. \tag{B.43}$$

and we remind the correspondence with boundary conditions:

$$\begin{aligned}
&a=b=0 \to A\underset{A}{\square}, && Z\left[{0 \atop 0}\right] = \mathrm{Tr}_{\mathrm{NS}} q^{L_0} = \sqrt{\frac{\vartheta_3}{\eta}},\\
&a=0,\, b=1 \to P\underset{A}{\square}, && Z\left[{0 \atop 1}\right] = q^{-1/48}\mathrm{Tr}_{\mathrm{NS}}(-1)^F q^{L_0} = \sqrt{\frac{\vartheta_4}{\eta}},\\
&a=1,\, b=0 \to A\underset{P}{\square}, && Z\left[{1 \atop 0}\right] = q^{-1/48}\mathrm{Tr}_{\mathrm{R}} q^{L_0} = \sqrt{\frac{\vartheta_2}{\eta}},\\
&a=1,\, b=1 \to P\underset{P}{\square}, && Z\left[{1 \atop 1}\right] = q^{-1/48}\mathrm{Tr}_{\mathrm{R}}(-1)^F q^{L_0} = \sqrt{\frac{\vartheta_1}{\eta}},
\end{aligned} \tag{B.44}$$

We now consider $\mathbf{Z}_2$ orbifolds of torus compactifications. For a boson on $\Gamma^{n,n}$, the orbifold blocks are given by

$$h\,\underset{g}{\square} \qquad Z\left[{h \atop g}\right] = 2^n \left|\frac{\eta}{\vartheta\left[{1-h \atop 1-g}\right]}\right|^n . \tag{B.45}$$

For fermions compactified on $\Gamma^{4,4}$, the orbifold blocks are given by

$$h\,\underset{g}{\square} \qquad Z\left[{b \atop a}\right]\left[{h \atop g}\right] = \frac{\vartheta\left[{a+h \atop b+g}\right]\vartheta\left[{a-h \atop b-g}\right]}{\eta^2} . \tag{B.46}$$

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