MODERN TRENDS IN STRING THEORY II

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M-Theory, Sasaki-Einstein Metrics, and New $\mathcal{N} = 1$ SCFTs

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What we do (and why)

- Develop a "systematics of string theory vacua"

- Understand general properties of backgrounds in an appropriate geometrical framework

- Provide a powerful method for constructing, i.e. finding explicitly, new supersymmetric solutions in string/M-theory

- "Flux compactifications"

- AdS/CFT correspondence and its cousins

- Mathematical spin-off

How do we do this?

• We start from scratch: analyze the conditions under which the Killing spinor equations (KSE) admit solutions $\longrightarrow$ preserved supersymmetry. Examples:

$$\begin{aligned} \nabla_\mu \epsilon = 0 \quad &\rightarrow \quad \text{special holonomy : Calabi} - \text{Yau, etc.} \\ \nabla_\mu \epsilon + \tfrac{i}{2}\gamma_\mu \epsilon = 0 \quad &\rightarrow \quad \text{AdS/spheres} + \text{Sasaki} - \text{Einstein} \end{aligned}$$

• We use the tool of "G-structures" (G= "group") [GMPW]. A G-structure consists of a number of geometric data (vector fields, forms) which obey differential conditions. These are then equivalent to the relevant KSE.

• Example: on a Riemannian manifold M_{2n} $\exists$ ϵ s.t.

$$\nabla_\mu \epsilon + \ iA_\mu \ \epsilon = 0$$

$$\Updownarrow$$

M_{2n} is Kähler $\equiv$ has an integrable $U(n)$ structure

What did we get?

• This talk is about $d = 11$ supergravity solutions with AdS_5 factor + by-products

1) Geometrical characterization of all supersymmetric solutions. Key point is to make contact with Kähler geometry

2) Plenty of explicit new solutions which are complex S^2 bundles over a four-dimensional Kähler space M_4:

(i) M_4 is Kähler-Einstein (KE)

(ii) $M_4 = \Sigma_1 \times \Sigma_2$ (non-Einstein!)

3) When $\Sigma_1 = T^2$: reduction + T-duality leads to IIA/B (classically/locally) supersymmetric solutions

4) Quantum/global analysis of new solutions. At the end of the day, we have: new 5d Sasaki-Einstein metrics (on $S^2 \times S^3$)

5) AdS/CFT dictionary $\Rightarrow \mathcal{N} = 1$ SCFT

6) The mathematical construction extends to any dimension. We construct infinitely many new SE metrics in arbitrary (odd!) dimension. In particular, in $d = 7$ new candidates for $\mathcal{N} = 2$ SCFT in 3d

Rules of the game

• 11-dim supergravity

Bosonic fields: g_{MN}, G_{MNPQ}

Killing spinor equation:

$$\delta\psi_M = \hat{\nabla}_M \eta - \frac{1}{288}\left(G_{NPQR}\hat{\Gamma}^{NPQR}{}_M - 8G_{MNPQ}\hat{\Gamma}^{NPQ}\right)\eta = 0$$

Bianchi identity: $\mathrm{d}G = 0$

G eq. of motion: $\mathrm{d}\,\hat{*}G + \frac{1}{2}G\wedge G = 0$

• We are interested in solutions of the type:

Warped metric $\quad d\hat{s}^2_{11} = \mathrm{e}^{2\lambda(x)}[ds^2(AdS_5) + ds^2(M_6)]$

Flux $\quad G$ arbitrary

Spinor $\quad \eta = \psi\otimes\xi$. ξ is a generic spinor, i.e. non-chiral

$$\nabla_\mu\psi = \frac{\mathrm{i}}{2}m\gamma_\mu\psi$$

• If ξ is a 6d chiral spinor

$\Rightarrow$ $G = 0$, $\lambda =$constant, M_6 is a Calabi–Yau three-fold, and $m = 0$ i.e. AdS_5 is not possible.

The local geometry on M_6

- ξ complex, non-chiral spinor $\Leftrightarrow$ local $SU(2)$-structure on M_6, characterized by

$$J,\ \Omega_{(2,0)},\ K^1,\ K^2,\ \sin\zeta \sim ||\xi||$$

- The metric on M_6 has the canonical form:

$$ds_6^2 = e^{-6\lambda}[ds^2(M_4) + \sec^2\zeta dy^2] + \frac{1}{9m^2}\cos^2\zeta(d\psi + \rho)^2$$

- $K^{2\#} = \sec\zeta\frac{\partial}{\partial\psi}$ is a Killing vector $\rightarrow U(1)$ R-symmetry

- At any fixed y, M_4 is Kähler:

$$\mathrm{d}_4 J = 0 \qquad \mathrm{d}_4\Omega = \mathrm{i}P\wedge\Omega$$

- "Dynamical" equations (evolution of the Kähler structure along y):

$$\begin{aligned} e^{3\lambda}\sin\zeta &= 2my \\ \rho &= P + \mathrm{i}(\partial - \bar\partial)\log\cos\zeta \\ \partial_y J &= -\tfrac{2}{3}y\mathrm{d}_4\rho \\ \partial_y \log\sqrt{g(M_4)} &= -3y^{-1}\tan^2\zeta - 2\partial_y\log\cos\zeta \end{aligned}$$

- The flux is fixed by the geometry:

$$e^{-3\lambda} *_6 G = \mathrm{e}^{-6\lambda}\mathrm{d}(\mathrm{e}^{6\lambda}\cos\zeta K^2) - 4m(J - K^1\wedge K^2\sin\zeta)$$

A very useful lemma!

Lemma: All the solutions of the type we are considering solve automatically the Bianchi identity $\mathrm{d}G = 0$ and the equations of motion.

• This is typically a major complication for using the formalism to construct actual solutions (because, at best, one is reduced to solve second-order non-linear PDE's!). Luckily, this is not the case here.

Plenty of new solutions

NB: in general we require (the metric on) M_6 to be compact, non-singular, and possibly new

• Now, by hand, we require that $ds^2(M_6)$ is a Hermitian metric on a complex manifold (wrt a natural complex structure inherited from J, Ω, K^1, K^2)

• Simplified local conditions on the geometry

$$\begin{aligned} \rho &= P \\ \mathrm{d}_4\lambda = \mathrm{d}_4\zeta = \partial_y P &= 0 \end{aligned}$$

• We now face a system of two coupled ODE's for the function $\lambda(y)$. If we solve it, we are done.

What's the global structure of the solutions?

• P is the connection on $\mathcal{L}$ (canonical line bundle of M_4)

• We would like to require the (y, ψ) coordinates to describe a smooth $\mathbb{C}P^1 = S^2$. Hence we have:

$$\begin{array}{ccc} \mathbb{C}P^1_{y,\psi} & \longrightarrow & M_6 \\ & & \downarrow \\ & & M_4 \end{array}$$

It turns out that M_6 is the bundle of unit *self-dual* two-forms on M_4:

$$\Lambda^+(M_4) \simeq \mathbb{R}[J] \oplus \Lambda^{(2,0)} \oplus \Lambda^{(0,2)} \simeq \mathcal{O} \oplus \mathcal{L}_{\mathbb{R}}$$

We find all solutions analytically

Lemma: at fixed y, the Ricci tensor on M_4 has two pairs of constant eigenvalues.

• Now, using a theorem in [math.DG/0007122], we conclude that (at fixed coordinate y) either

(i) M_4 is Kähler-Einstein (KE)

or

(ii) M_4 is $\Sigma_1 \times \Sigma_2$ with a non-Einstein metric

• We have now two very simple ansatze for the metrics on M_4. We consider them in turn and feed back into the remaining two ODE's for $\lambda(y)$.

• Miraculously, the system is integrable (!), and we obtain analytic solutions for λ and $ds^2(M_4)$ in both cases.

• The last step is to check when these correspond to smooth, global metrics on M_6.

[Recall that we don't have to worry at all about the flux.]

A summary of the non-singular solutions

(i) Kähler-Einstein base

What do the solutions look like? Here is a representative:

$$e^{6\lambda} = \frac{2m^2(k-y^2)^2}{cy+2k+2y^2}$$

c is a constant, $k = 0, \pm 1$ is the the curvature.

• For generic c, the solutions will have curvature or conical singularities. Can we fiddle with c to get something good?

• $[k = 1]$ For $0 \leq c < 4$ we have a one-parameter family of regular, compact, complex solutions with the topology of a $\mathbb{C}P^1$ fibration over a positive curvature KE space.

(ii) $\Sigma_1 \times \Sigma_2$ base

• $[k_1, k_2 = \pm 1]$ For various ranges of integration constants there are regular solutions that are topologically S^2 bundles over $S^2 \times S^2$ and $S^2 \times H^2$.

• $[k_1 = 0 : \Sigma_1 = T^2]$ For $0 < a < 1$ and $c \neq 0$ we have a one-parameter family of regular, compact, complex solutions that are S^2 bundles over $S^2 \times T^2$. A single additional solution of this type is obtained when $c = 0$ and $a \neq 0$.

Where

$$e^{6\lambda} = \frac{2m^2(a - ky^2)}{k - cy}$$

The dualized solutions in Type IIA/B

- We now focus on the "$T^2 \times S^2$ case"

- The solution in M-theory

$$\begin{aligned} ds^2 &= e^{2\lambda} ds^2(AdS_5) + e^{-6\lambda} \sec^2\zeta dy^2 \\ &+ \tfrac{1-cy}{6m^2} ds^2(S^2) + \tfrac{1}{9m^2}\cos^2\zeta (d\psi+\rho)^2 + e^{-6\lambda} ds^2(T^2) \\ \cos^2\zeta &= \frac{a-3y^2+2cy^3}{a-y^2} \\ G &= \text{not enlightening} \end{aligned}$$

Isometry: $SU(2) \times \underbrace{U(1) \times U(1)}_{T^2} \times U(1)_R$

- Reduction/T-duality along $U(1)_R$ breaks supersymetry

The classical example is "S^5 untwisted" [Duff,Lu,Pope]: $S^1 \to S^5 \to \mathbb{C}P^2$ T-duality on S^1 fibre gives $\mathbb{C}P^2 \times S^1$ - not even spin!

- However, reduction/T-duality along on T^2 is supersymmetric

Type IIA:

$$AdS_5 \times X_4 \times S^1$$

$$e^{2\Phi} = e^{-6\lambda},\ G^{(4)}_{RR},\ H^{(3)}_{NS} = dB$$

Type IIB:

$$
\begin{aligned}
ds^2 &= \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 \\
&+ \frac{q(y)}{9}[d\psi - \cos\theta d\phi]^2 + w(y)\,[d\alpha + A]^2
\end{aligned}
$$

with

$$
\begin{aligned}
w(y) &= \frac{2(a-y^2)}{1-cy} \\
q(y) &= \frac{a-3y^2+2cy^3}{a-y^2} \\
A &= \frac{ac-2y+y^2c}{6(a-y^2)}[d\psi - \cos\theta d\phi]
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\Phi} &= \text{constant} \\
F_{RR}^{(5)} &= (1+*)\text{vol}(AdS_5)
\end{aligned}
$$

• In summary: we have found a supersymmetric two-parameter family of $AdS_5 \times X_5$ solutions of Type IIB supergravity. These must be Sasaki-Einstein (SE). So, what are they?

Sasaki-Einstein geometry (short summary)

• The handiest definition of a (necessarily odd-dimensional) SE manifold is that its "metric cone"

$$ds^2_{\mathcal{C}} \;=\; dr^2 + r^2 ds^2(\mathrm{SE})$$

be Kähler (S) and Ricci-flat (E) = a Calabi-Yau cone

Everyone is familiar with:

(i) S^5 whose cone is $\mathbb{R}^6$ (ii) T^{11} whose cone is the "conifold"

• The geometric data of a SE structure are

∘ a unit-norm Killing vector $V = \frac{\partial}{\partial\psi}$

∘ its dual one-form η

∘ a two-form J_4 such that $\mathrm{d}\eta = 2J_4$

Killing spinors

• A (simply connected and spin) SE manifold admits spinors satisfying

$$\nabla_\mu \xi \;=\; \pm\tfrac{i}{2}\gamma_\mu \xi$$

• That's why $AdS_5\times$SE is supersymmetric in Type IIB

"Transverse" geometry

• V defines (locally) a "foliation". I.e.

$$ds^2(\mathrm{SE}) = (\mathrm{d}\psi + \eta)^2 + \mathrm{d}s_4^2$$

$(\mathrm{d}s_4^2, J_4)$ is (at least locally) Kähler-Einstein

• We have a finer classification of SE manifolds

1) *Regular*: $(\mathrm{d}s_4^2, J_4)$ is a KE manifold

2) *Quasi-regular*: $(\mathrm{d}s_4^2, J_4)$ is a KE orbifold

3) *Irregular*: $(\mathrm{d}s_4^2, J_4)$ is not globally defined = "badly singular"

• Only in cases 1) and 2) can the SE be defined as $U(1)$ bundle over the respective KE base manifold/orbifold.

• In case 3) this operation doesn't make sense, as the orbits of the KV don't close

A look at what's known

• Regular SE: completely classified

KE four-manifolds: $S^2 \times S^2$, $\mathbb{C}P^2$, del Pezzo P_k $3 \leq k \leq 8$

(i) $S^1 \to S^5 \to \mathbb{C}P^2 \ , \quad \mathbb{C}P^2 = S^5/U(1)$

(ii) $S^1 \to T^{11} \to S^2 \times S^2 \ , \quad S^2 \times S^2 = T^{11}/U(1)$

(iii) del Pezzo's (metrics unknown)

• Quasi-regular SE: recently constructed using algebraic-geometry techniques.
Examples on $l\#(S^3 \times S^2)$. The metrics are not known.

• Irregular SE: there are no examples.

How do our metrics fit into this picture?

The new solutions

• We address the problem of identifying the $Y^{(a,c)}$ spaces in two steps

(I) local analysis (easy)

(II) global analysis (tricky)

Local analysis

• A change of coordinates makes manifest the expected SE structure [c rescaled to 1. I'll comment later on $c = 0$]

$$
\begin{aligned}
ds^2 &= ds_4^2 + \left(\tfrac{1}{3}d\psi' + \sigma\right)^2 \\
ds_4^2 &= \frac{1}{\Delta}d\rho^2 + \frac{\rho^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \Delta\sigma_3^2\right) \\
\Delta &= 1 + \frac{4(a-1)}{27}\frac{1}{\rho^4} - \rho^2
\end{aligned}
$$

ds_4^2 is known to be a (local) Kähler-Einstein metric [Gibbons+Pope (1979)]

• In fact, it is known that ds_4^2 is singular (except for $a = 1$ when it is the standard Einstein metric on $\mathbb{C}P^2$). These coordinates aren't helpful for analyzing the global structure of $Y^{(a,c)}$

Global analysis

▶ We use the "M-theory coordinates"

$$ds^2 \;=\; ds^2(B_4) + w(y)(d\alpha + A)^2$$

- Strategy

1) Show that B_4 is a smooth manifold

2) Show that A can be made a connection on a $U(1) \to Y^{(a,c)} \to B_4$ bundle for carefully chosen values of a

3) Identify the topology of $Y^{(a,c)}$

1) The base B_4

$$ds^2(B_4) = \frac{1-y}{6}(\sigma_1^2 + \sigma_2^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}\sigma_3^2$$

- For $0 < a < 1$ this is a non-singular metric on an S^2 bundle over S^2

- $c_1 = \frac{1}{2\pi}\int_{\text{base } S^2} d\sigma_3 = 2$ even $\Rightarrow$ the bundle is trivial:

$$B_4 \;=\; S^2 \times S^2$$

2) The circle fibration

$A = \frac{a-2y+y^2}{6(a-y^2)}\sigma_3$ is a connection on a $U(1)$ bundle if

$$P_i = \frac{1}{2\pi}\int_{S_i^2} dA = \ell p, \ell q$$

for some integers p, q.

- P_i are functions of a, and it turns out that

$$\frac{P_1}{P_2} = \frac{p}{q}$$

is satisfied for a countably infinite number of values of $0 < a < 1$

$\alpha \in [0, 2\pi\ell]$ with $\ell = \frac{q}{3q^2-2p^2+p(4p^2-3q^2)^{1/2}}$

The canonical constant-norm KV V is

$$V = \frac{\partial}{\partial\psi'} = \frac{\partial}{\partial\psi} - \frac{1}{6}\frac{\partial}{\partial\alpha}$$

Clearly, if ℓ is irrational the orbits of V don't close $\Rightarrow$ the SE is irregular

On the contrary, if ℓ is rational we can quotient by the $U(1)$ associated to V, hence the SE is quasi-regular

The volume of $Y^{p,q}$ is given by

$$\text{vol}(Y^{p,q}) = \frac{q^2[2p+(4p^2-3q^2)^{1/2}]}{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}\pi^3$$

3) The global topology

It turns out that, topologically:

$$Y^{p,q} = S^2 \times S^3$$

[We show that chosing p, q co-prime, they are simply-connected and use a theorem of Smale [Smale (1962)]]

• Special cases. By direct inspection we learn that

(i) $c = 0$: $Y^{(a,0)} \equiv T^{11}$

(ii) $a = 1$: $Y^{(1,c)} \equiv S^5$

• Remark: recall that we showed the metrics were perfectly smooth and well behaved in Type IIA & M theories. Is this is a puzzle? No!

It is a good example of how T-duality exchanges classical with quantum properties:

II B: global metric on $Y^{p,q}$ $\Leftrightarrow$ IIA&M: flux quantization

Summary

We have a family of infinitely many new explicit SE metrics $Y^{p,q}$ on $S^2 \times S^3$ with $SU(2) \times U(1) \times U(1)$ isometry, with p, q co-prime integers.

They are generically *irregular*, but include also infinitely many *quasi-regular* metrics, when $4p^2 - 3q^2 = n^2$.

The metrics "interpolate" between the two limiting cases of S^5 and $T^{11} \simeq S^2 \times S^3$.

AdS/CFT and $\mathcal{N} = 1$ SCFTs

AdS/CFT: $AdS_5 \times X_5$, with X_5 SE should be dual, in the large N limit, to an $\mathcal{N} = 1$ SCFT, arising on N $D3$ branes placed at the tip of the Calabi-Yau cone $\mathcal{C}(X_5)$.

The isometries of X_5 correspond to "flavor" symmetries of the SCFT. More precisely, the symmetry group is

$$\underbrace{SU(2,2|1)}_{\supset U(1)_R} \times \underbrace{\mathcal{F}}_{\text{non}-R}$$

- Central charges $c = a \sim \frac{1}{\text{vol}(X_5)}$

▷ The simplest example is the "conifold"

[Klebanov,Witten]

∘ Isometry of T^{11}: $SU(2) \times SU(2) \times U(1)_R$

∘ The SCFT is $SU(N) \times SU(N)$ with chiral fields in bi-fundamental + a superpotential W

∘ $c = \frac{27}{16} c(\mathcal{N} = 4)$

A field theory slide

To appreciate our results in the context of AdS/CFT, I have to summarize recent results on general $\mathcal{N} = 1$ SCFT in $d = 4$ due to [Intriligator,Wecht].

• They presented a general method for determining uniquely the R-symmetry of an $\mathcal{N} = 1$ SCFT

• Formulas for central charges [Anselmi et al.]

$$a = \tfrac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R) \ , \qquad c = \frac{1}{32}(9\mathrm{Tr}R^3 - 5\mathrm{Tr})$$

• Consider a "trial" R-symmetry $R_t = R_0 + \sum_I s^I F_I$

∘ R_0 is a possible R-symmetry, i.e. obeys some consistency requirements

∘ F_I are generators of $U(1)$'s $\subset \mathcal{F}$

∘ $s^I \in \mathbb{R}$ (a priori)

Result: the exact R-symmetry is the one which *maximazes* the central charge a. In particular it's determined by

$$9\mathrm{Tr}(R^2 F_I) = \mathrm{Tr}F_I \ , \qquad \mathrm{Tr}(R F_I F_K) < 0$$

• A corollary of this result is that $\mathcal{N} = 1$ SCFTs are "*algebraic* ", i.e. their R- and central charges are square roots of rational numbers

Example given in [IW]: quiver $U(N)^4$ with $W = 0$

$$a \ = \ \tfrac{3+5\sqrt{5}}{16} N^2$$

The same theory with suitable W is dual to $T^{11}/\mathbb{Z}_2$, and $a = \frac{27}{32}N^2$!

The $\mathcal{N} = 1$ SCFTs dual to $Y^{p,q}$

• Isometry: $SU(2) \times U(1)_{\mathcal{F}} \times U(1)_R$

Geometry tells us who is $U(1)_R \subset U(1) \times U(1)$. Recall the R–symmetry generator = SE canonical KV is

$$\frac{\partial}{\partial \psi'} = \frac{\partial}{\partial \psi} - \frac{\ell}{6}\frac{\partial}{\partial \alpha'}$$

with $(\psi, \alpha') \in [0, 2\pi] \times [0, 2\pi]$, and

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}$$

• The central charges are

$$c^{-1}(Y^{p,q}) \quad \sim \quad \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}$$

► c, ℓ (hence the R-charges) are *algebraic* numbers

Geometry: $Y^{p,q}$ is *irregular* $\Leftrightarrow$ its volume is *irrational*

$\Updownarrow$ $\Updownarrow$

SCFT: R-charges irrational $\Leftrightarrow$ c, a irrational

• There are many unanswered questions: gauge group, superpotential, ...

These are related to open questions regarding the Calabi-Yau cones $\mathcal{C}(Y^{p,q})$

What do we do next?

☐ Identify SCFT's exactly → many things to do

☐ Derive a "gravity dual" of a-maximization

☐ Investigate the new Calabi-Yau cones

☐ Deform/resolve CY (as for the conifold)

☐ Study the physics (SCFT) of the remaining solutions which we presented

☐ Find further AdS_5 solutions in M-theory: non-complex M_6

☐ Construct new KE + SE metrics (now that we know the trick)