Gauged Six-Dimensional Chiral Supergravity and its Origins from String Theory

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6-Dimensional Chiral Gauged Supergravity

There is a well-known non-chiral $\mathcal{N} = (1,1)$ gauged supergravity in six dimensions (Romans), which arises from a consistent (warped) S^4 reduction of the massive type IIA theory in ten dimensions. This has a negative (AdS-type) scalar potential. The theory cannot be truncated to chiral $\mathcal{N} = (1,0)$ supergravity.

There exist also inequivalent *chiral* $\mathcal{N} = (1,0)$ gauged supergravities in $D = 6$, for which the scalar potential has the opposite sign. (Sezgin/Nishino, Salam/Sezgin.)

The simplest example is the "Salam-Sezgin theory," which is a gauging of pure $\mathcal{N} = (1,0)$ supergravity coupled to one vector multiplet and one tensor multiplet.

The bosonic Lagrangian is

$$\mathcal{L} = R - \tfrac{1}{4}(\partial\phi)^2 - \tfrac{1}{12}e^{\phi}\, H_{(3)}^2 - \tfrac{1}{4}e^{\frac{1}{2}\phi}\, F_{(2)}^2$$
$$-8g^2\, e^{-\frac{1}{2}\phi}$$

where g is the gauge-coupling constant.

The theory has a $(\text{Minkowski})_4 \times S^2$ vacuum, with

$$ds_6^2 = e^{\frac{1}{2}\phi_0}\, \eta_{\mu\nu}\, dx^\mu\, dx^\nu + \frac{1}{8g^2}\, e^{-\frac{1}{2}\phi_0}\, d\Omega_2^2$$
$$F_{(2)} = \frac{1}{4g}\, e^{\frac{1}{2}\phi_0}\, \Omega_{(2)}\,, \quad \phi = \phi_0 = \text{const.}$$

The positive scalar potential balances the negative contribution of from the Freund-Rubin term. The dilaton provides a "self-tuning" field that selects Minkowski space-time.

$(\text{Minkowski})_4 \times S^2$ is the vacuum of a consistent S^2 Pauli reduction, yielding an $\mathcal{N} = 1$ four-dimensional supergravity.

Higher-Dimensional Origins

Cvetič, Gibbons, Pope

A sphere reduction gives a negative cosmological constant, so to get Salam-Sezgin from a higher dimension, we need a different type of reduction. First, we shall discuss how it arises from an S^1 reduction of a non-compact $SO(2,2)$ gauged supergravity in $D=7$.

$$\begin{aligned}
\mathcal{L}_7 &= R*\mathbf{1} - \tfrac{5}{16}\Phi^{-2}*d\Phi\wedge d\Phi - *p_{\alpha\beta}\wedge p^{\alpha\beta} \\
&\quad -\tfrac{1}{2}\Phi^{-1}*H_{(3)}\wedge H_{(3)} - V*\mathbf{1} \\
&\quad -\tfrac{1}{2}\Phi^{-1/2}\pi_A{}^\alpha\pi_B{}^\beta\pi_C{}^\alpha\pi_D{}^\beta *F_{(2)}^{AB}\wedge F_{(2)}^{CD}
\end{aligned}$$

where

$$\begin{aligned}
V &= \tfrac{1}{2}g^2\Phi^{1/2}\,[2M_{\alpha\beta}M_{\alpha\beta} - (M_{\alpha\alpha})^2] \\
M_{\alpha\beta} &= \pi^{-1}{}_\alpha{}^A\pi^{-1}{}_\beta{}^B\eta_{AB}
\end{aligned}$$

Nine scalars described by the "vielbein" $\pi^\alpha{}_A$. The indices α are $SO(4)_c$; the indices A are $SO(4,m-4)_g$, with metric η_{AB}.

In the "vacuum" with $\pi^\alpha{}_A = \delta^\alpha_A$, the compact gauging $SO(4)_g$ has $V = -4g^2\Phi^{1/2}$, but the non-compact $SO(2,2)_g$ gauging has $V = +4g^2\Phi^{1/2}$.

As well as providing the right sign for the potential, the non-compact gauging is also essential for allowing a chiral truncation upon reduction to $D = 6$. This can be illustrated in the $D = 7$ gravitino transformation rule

$$\delta\psi_\mu = D_\mu \epsilon - \tfrac{1}{20} g M_{\alpha\alpha}\, \Phi^{1/4}\, \Gamma_\mu\, \epsilon + (F, H)_\mu \epsilon$$

The term involving $M_{\alpha\alpha}$ reduces to give a chirality-reversing term in $D = 6$. But in the non-compact gauging, if $\pi^\alpha{}_A = \delta^\alpha_A$, we have $M_{\alpha\alpha} = 0$, and hence chiral truncation consistent.

Other "conspiracies" occur too, and a fully consistent and supersymmetric reduction to give the Salam-Sezgin theory is possible. This involves setting all $A_\mu^{AB} = 0$ except $A_\mu^{12} = -A_\mu^{34} = \frac{1}{2} A_\mu$ (the $U(1)$ gauge field of Salam-Sezgin), and setting the Kaluza-Klein vector to zero.

Reminiscent of the Z_2 truncation in Horava-Witten.

The seven-dimensional $SO(2,2)$ gauged supergravity can itself be obtained via a consistent reduction from $D=10$, $\mathcal{N}=1$ supergravity. It can be viewed as a limiting case of the (non-compact version of) the consistent S^4 reduction from $D=11$.

$$\begin{aligned} d\hat{s}_{10}^2 &= \Phi^{\frac{3}{16}}\Delta(ds_7^2 + \tfrac{1}{2}g^{-2}\Delta^{-1}M_{AB}^{-1}D\mu^A D\mu^B) \\ \hat{F}_{(3)} &= \cdots \\ e^{\hat{\phi}} &= \Phi^{\frac{5}{8}}\Delta^{-\frac{1}{2}} \end{aligned}$$

where

$$\begin{aligned} D\mu^A &= d\mu^A + 2gA^A{}_B\,\mu^B\,, \quad \Delta = M_{AB}\,\mu^A\,\mu^B \\ M_{AB} &= \pi^\alpha{}_A\,\pi^\alpha{}_B\,, \quad \eta_{AB}\,\mu^A\mu^B = 1 \end{aligned}$$

The four coordinates μ^A subject to

$$\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$$

define the 3-dimensional hyperboloid $\mathcal{H}^{2,2}$, embedded in Euclidean space E^4 with the standard metric

$$ds^2 = d\mu_1^2 + d\mu_2^2 + d\mu_3^2 + d\mu_4^2$$

This has isometry group $SO(2)\times SO(2) = SO(2,2)\cap SO(4)$.

Ghost-Free Non-Compact Gaugings

The scalar fields M_{AB} play an essential rôle in the non-compact gauging.

If there were no scalars in the theory, the only way to have $SO(2,2)$-invariant kinetic terms would be to contract $F_{\mu\nu}^{AB}\, F^{CD\,\mu\nu}$ with the (indefinite-signature) Cartan-Killing metric of $SO(2,2)$. This would imply wrong-sign kinetic terms, and hence ghosts.

In any ground state (constant scalars), the non-compact gauge group is spontaneously broken to a compact subgroup. This is reflected in the metric $ds^2 = M_{AB}\, D\mu^A\, D\mu^B$ on the internal space, which has only $SO(2)\times SO(2)$ isometry in the $\pi^\alpha{}_A = \delta^\alpha_A$ vacuum.

Although the reduction from $D = 10$ to $D = 7$ on $\mathcal{H}^{2,2}$ is really just an analytic continuation of the S^3 reduction that gives an $SO(4)$ gauged supergravity, the nature of the "internal space" is very different: S^3 is a homogeneous space, while $\mathcal{H}^{2,2}$ is not.

$D = 10$ to $D = 6$ Reduction

Solving $\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$ by writing

$$\mu_1 + i\,\mu_2 = \cosh\rho\, e^{i\alpha}\,, \qquad \mu_3 + i\,\mu_4 = \sinh\rho\, e^{i\beta}$$

the metric on $\mathcal{H}^{2,2}$ is

$$ds_3^2 = \cosh 2\rho\, d\rho^2 + \cosh^2\rho\, d\alpha^2 + \sinh^2\rho\, d\beta^2$$

Combining the $D = 10$ to $D = 7$ reduction with the further S^1 reduction to the Salam-Sezgin theory, the reduction ansatz is

$$\begin{aligned} d\hat{s}_{10}^2 &= (\cosh 2\rho)^{\frac{1}{4}}\{e^{-\frac{1}{4}\phi} ds_6^2 + e^{\frac{1}{4}\phi} dz^2 \\ &\quad + \tfrac{1}{2} g^2 e^{\frac{1}{4}\phi}[d\rho^2 + \frac{\cosh^2\rho}{\cosh 2\rho}\,(d\alpha - gA)^2 \\ &\quad + \frac{\sinh^2\rho}{\cosh 2\rho}\,(d\beta + gA)^2]\} \\ \hat{F}_{(3)} &= \cdots \\ e^{\hat{\phi}} &= (\cosh 2\rho)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\phi} \end{aligned}$$

This allows any solution of the six-dimensional Salam-Sezgin theory to be lifted back to an exact solution of ten-dimensional $\mathcal{N} = 1$ supergravity.

Uniqueness of (Minkowski)$_4 \times S^2$ Vacuum

Gibbons, Güven, Pope

Consider a more general configuration in the Salam-Sezgin theory with maximal four-dimensional symmetry:

$$
\begin{aligned}
ds_6^2 &= W(y)\, ds_4^2 + g_{mn}\, dy^m\, dy^n\,, \qquad \phi = \phi(y) \\
F_{mn} &= f(y)\, \epsilon_{mn}\,, \qquad F_{\mu\nu} = F_{\mu n} = 0 \qquad H_{(3)} = 0
\end{aligned}
$$

where ds_4^2 satisfies $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$ and has maximal symmetry (Minkowksi, AdS or dS). From the Einstein and dilaton equations we find

$$\nabla^m (W^4\, \nabla_m(\phi - 4 \log W)) + 4\Lambda\, W^2 = 0\,. \quad (1)$$

Integrating over the (compact) internal 2-space Y with metric $g_{mn} dy^m dy^n$ implies

$$\Lambda \int_Y W^2 = 0\,, \qquad \text{hence } \Lambda = 0$$

and thus Minkoswki$_4$ spacetime. Then multiplying (1) by $(\phi - 4\log W)$ and integrating implies $\phi = 4 \log W$. The tracefree part of the R_{mn} equation implies $\nabla_m \nabla_n W^2 = \frac{1}{2} \nabla^2 W^2 g_{mn}$ and hence $K^m = \epsilon^{mn}\, \nabla_n W^2$ is a Killing vector on Y.

The field equation for $F_{(2)}$ implies

$$F_{(2)} = \tfrac{1}{2} q\, W^{-6}\, \epsilon_{mn} dy^m \wedge dy^n$$

carries magnetic charge q. The trace of the R_{mn} equation implies that the Gauss curvature κ of Y (i.e. $R_{mn} = \kappa g_{mn}$) integrates to give

$$\begin{aligned} \chi \;&=\; \tfrac{1}{2\pi} \int_Y \kappa \\ &= \tfrac{1}{2\pi} \int_Y [\frac{4(\nabla W)^2}{W^2} + \tfrac{3}{8} q^2 W^{-10} + 2 g^2 W^{-2}] \end{aligned}$$

This shows Y has positive Euler number, and since, by assumption, it is complete, compact and non-singular, it must be topologically S^2. The Killing vector K^m then must have circular orbits with 2 fixed points (i.e. azimuthal symmetry).

We can therefore write the metric as

$$ds_6^2 = W(\rho)^2\, \eta_{\mu\nu}\, dx^\mu dx^\nu + d\rho^2 + a(\rho)^2\, d\psi^2$$

and the remaining ODEs can be solved exactly to give

Axisymmetric Solutions

$$ds_2^2 = e^{\frac{1}{2}\phi}\Big(\frac{dr^2}{f_0^2} + \frac{r^2}{f_1^2}\,d\psi^2\Big)$$
$$F_{(2)} = \frac{q\,r\,e^{-\phi}}{f_0\,f_1}\,dr\wedge d\psi$$
$$e^{-\phi} = \frac{f_0}{f_1}\,,\qquad f_0 = 1+\frac{r^2}{r_0^2}\,,\qquad f_1 = 1+\frac{r^2}{r_1^2}$$

where

$$r_0^2 = \frac{1}{2g^2}\,,\qquad r_1^2 = \frac{8}{q^2}$$

The metric is regular only if $r_0 = r_1$ (i.e. $q = \pm 4g$), in which case $W = 1$, $\phi = 0$ and, setting $r = r_0 \tan\frac{1}{2}\theta$, we recover the Salam-Sezgin $(\text{Minkowski})_4\times S^2$ vacuum with metric

$$ds_2^2 = \tfrac{1}{4}r_0^2\,(d\theta^2 + \sin^2\theta\,d\psi^2)$$

on the internal space Y. This is therefore the only non-singular solution with a four-dimensional maximally-symmetric spacetime.

3-Brane Solutions

The more general axisymmetric solutions, with $r_1 \neq r_0$, describe configurations with conical curvature singularities at one or both "poles" of the 2-sphere, at $r = 0$ and $r = \infty$. Choosing ψ to have period 2π, the internal space Y is smooth at $r = 0$, but has a conical singularity with deficit angle

$$\delta = 2\pi\,(1 - \frac{r_1^2}{r_0^2})$$

at $r = \infty$. The 3-brane tension is positive ($\delta > 0$) if $r_0 > r_1$ and negative if $r_0 < r_1$. Dirac quantisation of the magnetic charge q implies

$$\frac{4g}{q} = N = \text{integer}$$

and hence

$$\delta = 2\pi(1 - N^2) \leq 0$$

Thus the 3-brane tension is negative.

Pauli Reductions

Gibbons, Pope

Pauli (1953) was the first to propose that one might obtain non-abelian gauge fields from the isometries of an internal manifold in a dimensional reduction. His *Ur* example was a reduction on S^2 giving $SU(2)$ Yang-Mills. However, such a reduction on a coset manifold will not, in general, be consistent with the higher-dimensional equations of motion. For example, reducing pure Einstein gravity on a space with Killing vectors K_I^m would give a lower-dimensional "Einstein equation" of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2} g_{mn} K_I^m K_J^n (F_{\mu\rho}^I F_\nu^{J\,\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^I F^{J\,\rho\sigma})$$

which doesn't make sense because the prefactor on the right-hand side depends on the coordinates y^m of the internal space.

This problem can however be evaded in special theories, notably certain supergravities, where contributions from additional fields cancel the undesirable y-dependence.

Examples are the S^7 and S^4 reductions of $D = 11$ supergravity, and the S^5 reduction of type IIB supergravity. However, the simplest case, realising Pauli's *Ur* example, is the S^2 reduction of the Salam-Sezgin theory. The bosonic reduction ansatz is

$$\begin{aligned} d\hat{s}_6^2 &= e^{\frac{1}{2}\phi} ds_4^2 \\ &\quad + e^{-\frac{1}{2}\phi} g_{mn}(dy^m + gA^i K_i^m)(dy^n + gA^j K_j^n) \\ \hat{H}_{(3)} &= H_{(3)} - 2g e^{\frac{1}{4}\phi} F^i \wedge K_i^a \hat{e}^a \\ \hat{F}_2 &= 2g e^{\frac{1}{2}\phi} \epsilon_{ab} \hat{e}^a \wedge \hat{e}^b - \mu^i F^i \\ \hat{\phi} &= -\phi \end{aligned}$$

where $\hat{e}^a = e^a + gA^i K_i^a$. The three coordinates μ^i satisfy $\mu^i \mu^i = 1$ and parameterise the internal 2-sphere, whose Killing vectors are $K_i^m = (8g^2)^{-1} \epsilon^{mn} \partial_n \mu^i$.

The four-dimensional $SU(2)$ gauge bosons A_μ^i enter in the ansätze for $\hat{F}_2$ and $\hat{H}_{(3)}$ as well as $d\hat{s}_6^2$, and these extra contributions in the six-dimensional Einstein equations imply that one gets consistent four-dimensional equations of motion.

The Four-Dimensional Theory

Substitution of the ansatz into the six-dimensional field equations yields consistent four-dimensional equations derivable from the Lagrangian

$$\mathcal{L}_4 = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{-\phi}(F^i)^2 - \tfrac{1}{12}e^{-2\phi}H_3^2$$

The fermions reduce consistently too, yielding a four-dimensional $\mathcal{N}=1$ theory comprising supergravity coupled to an $SU(2)$ Yang-Mills multiplet and a scalar multiplet. The 3-form $H_{(3)}$ can be dualised to an axion σ, so that

$$\begin{aligned}\mathcal{L}_4 &= R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}(\partial\sigma)^2 \\ &\quad -\tfrac{1}{4}e^{-\phi}(F^i)^2 + \tfrac{1}{4}\sigma(F^i\cdot *F^i)\end{aligned}$$

The theory is non-chiral.

The complete four-dimensional reduction will also have infinite towers of massive fields, whose mass scale is given by

$$M \sim g\, e^{\frac{1}{2}\phi_0}$$

Non-Abelian pp-waves

Cariglia, Gibbons, Güven, Pope

The four-dimensional theory admits supersymmetric non-Abelian pp-waves, generalising solutions found by Coleman in flat spacetime. They take the form

$$
\begin{aligned}
ds_4^2 &= 2du\, dv + H(u,z,\bar z)\, du^2 + dz\, d\bar z \\
A^i &= \mathcal{A}^i(u,z,\bar z)\, du
\end{aligned}
$$

with the dilaton ϕ and axion σ being arbitrary functions of u. The $SU(2)$ Yang-Mills potentials and the function H are given by

$$
\begin{aligned}
\mathcal{A}^i &= \tfrac{1}{2}[\chi^i(u,z) + \bar\chi^i(u,\bar z)] \\
H &= K(u,z) + \bar K(u,\bar z) \\
&\quad -\tfrac{1}{4}[e^{-\phi}\chi^i\,\bar\chi^i + (\dot\phi^2 + e^{2\phi}\dot\sigma^2)\, z\bar z]
\end{aligned}
$$

where $\chi^i(u,z)$ and $K(u,z)$ are arbitary functions holomorphic in z.

Anomaly-Free $D = 6$ Chiral Supergravities

Bergshoeff, Pope, Sezgin, Stelle *in progress*

Six-dimensional chiral theories are subject to gravitational, gauge and mixed anomalies, and the Salam-Sezgin theory itself is anomalous.

The procedure that we used in reducing on S^1 from $D = 7$ to $D = 6$ involved a chiral truncation. In fact it was at this stage that the anomaly was introduced. It is closely analogous to the procedure whereby one can obtain the (anomalous) $\mathcal{N} = 1$ supergravity in ten dimensions by reduction of $D = 11$ supergravity on S^1, combined with a chiral truncation (i.e. a Kaluza-Klein reduction on S^1/Z_2).

The cure for the anomaly in that case is to invoke the Horava-Witten mechanism, with anomaly inflow on the 9-brane surfaces at endpoints of the S^1/Z_2 interval being provided by the introduction of $E_8 \times E_8$ gauge fields.

Thus the anomalous theory obtained in a classical Kaluza-Klein reduction is rendered anomaly-free by brane contributions in the full M-theory picture, leading to the $E_8 \times E_8$ heterotic string in $D = 10$.

The chiral truncation that we performed in order to obtain the Salam-Sezgin theory was actually more restrictive than a simple Z_2 factoring. In a full discussion at the level of M-theory and string theory, we need to perform a strict Z_2 factoring (yielding more fields classically than in the truncation to Salam-Sezgin). Additionally, we need to introduce appropriate gauge fields on the 5-branes at the endpoints of the S^1/Z_2 interval, together with Green-Schwarz anomaly-cancelling terms, in order to achieve an analogue of the Horava-Witten mechansim.

By this means, one should be able to arrive at an anomaly-free six-dimensional theory, whose low-energy limit will be contained within the more general class of chiral supergravities constructed by Nishino and Sezgin.