Rotating Kerr-de Sitter Black Holes In Higher Dimensions

Chris Pope
Texas A&M University

Modern Trends in String Theory II
Porto, 21–26 June 2004

Uncharged Black Holes in Arbitrary Dimension
In collaboration with:
H. Lü, G.W. Gibbons and D.N. Page

Charged Black Holes in Five Dimension
In collaboration with:
M. Cvetič and H. Lü

Kerr-de Sitter Black Holes

Four-dimensional rotating black holes were studied extensively in the 1960's. The general such solution of the Einstein equations $R_{\mu\nu} = 3\lambda\, g_{\mu\nu}$ has two parameters; the mass M and the angular momentum a.

In dimension D, the general rotating black hole has $[(D-1)/2]$ independent rotatation parameters, describing rotations in $[(D-1)/2]$ orthogonal 2-planes. The general such solution to the vacuum Einstein equations $R_{\mu\nu} = 0$ was found by Myers and Perry (1986).

The rotating black holes with cosmological constant are of particular interest for the AdS/CFT correspondence, for studying the case with rotating boundary. The general Kerr-de Sitter metric in $D = 5$ (with 2 rotation parameters) was found by Hawking, Hunter and Taylor-Robinson (1998). They also found the Kerr-de Sitter metrics in arbitrary dimension D, in the special case that all $[(D-1)/2]$ rotation parameters a_i are set equal.

Kerr-Schild Metrics

Solving the Einstein equations for the general Kerr-de Sitter metrics from first principles is very difficult. An easier approach is to make an inspired guess, based on the structures seen in the known special cases of $D=4$ and $D=5$, and in arbitrary dimension with $\lambda=0$. However, testing that the guessed metric indeed satisfies $R_{\mu\nu}=(D-1)\,\lambda\, g_{\mu\nu}$ is also quite tricky. A great simplification can be achieved by writing the metric in Kerr-Schild form.

A Kerr-Schild metric takes the form

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}\,, \qquad h_{\mu\nu} \equiv f\, k_\mu\, k_\nu$$

where k_μ is a null geodesic vector with respect to the "fiducial" metric $g_{\mu\nu}$:

$$k^\mu\, k_\mu = 0\,, \qquad k^\mu\, \nabla_\mu\, k_\nu = 0$$

Clearly, the inverse metric $\widehat{g}^{\mu\nu}$ is given exactly by

$$\widehat{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}$$

Remarkably, the Ricci tensor is given *exactly* by

$$\begin{aligned} \widehat{R}^\mu{}_\nu &= R^\mu{}_\nu - h^\mu{}_\rho\, R^\rho{}_\nu + \tfrac{1}{2}\nabla_\rho\nabla_\nu h^{\mu\rho} \\ &\quad + \tfrac{1}{2}\nabla^\rho\nabla^\mu h_{\nu\rho} - \tfrac{1}{2}\nabla^\rho\nabla_\rho h^\mu{}_\nu \end{aligned}$$

In other words, the "linearised approximation" is exact.

This can provide a relatively simple way of verifyng that $\widehat{g}_{\mu\nu}$ satisfies the Einstein equations, provided that the fiducial metric $g_{\mu\nu}$ is not too complicated.

The previously-known $D = 4$ and $D = 5$ Kerr-de Sitter metrics can be written in Kerr-Schild form with $g_{\mu\nu}$ being the de Sitter metric. It is therefore natural to expect that this should be possible in all dimensions.

$D=4$ **Schwarzschild in Kerr-Schild Form**

As a simple example, consider the Schwarzschild metric

$$ds^2 = -F\,dt^2 + \frac{dr^2}{F} + r^2 d\Omega_2^2\,, \quad F \equiv 1 - \frac{2M}{r}$$

Rewriting this as

$$ds^2 = -F\,(dt - F^{-1}dr)(dt + F^{-1}dr) + r^2 d\Omega_2^2$$

and defining $du \equiv dt - F^{-1}\,dr$ gives

$$ds^2 = -du^2 - 2du\,dr + r^2 d\Omega_2^2 + \frac{2M}{r}\,du^2$$

A final redefinition $\tilde{t} = u + r$ gives

$$ds^2 = -d\tilde{t}^2 + dr^2 + r^2 d\Omega_2^2 + \frac{2M}{r}\,(d\tilde{t} - dr)^2$$

which is in Kerr-Schild form, written with Minkowski spacetime as the fiducial metric, and $k = d\tilde{t} - dr$ as the geodesic null 1-form.

de Sitter in Spheroidal Coordinates

The required fiducial metric for Kerr-de Sitter is de Sitter spacetime, but written in an unusual system of spheroidal coordinates. The discussion is a bit different according to whether the dimension is even or odd; consider $D = 2n+1 =$odd here. First, we write de Sitter as

$$ds^2 = -(1-\lambda y^2)\,dt^2 + \frac{dy^2}{1-\lambda y^2} + y^2\,d\Omega_{2n-1}^2$$

Next, write the S^{2n-1} metric as

$$d\Omega_{2n-1}^2 = \sum_{k=1}^{n} (d\hat\mu_k^2 + \hat\mu_k^2\,d\phi_k^2)$$

where

$$\sum_{k=1}^{n} \hat\mu_k^2 = 1$$

Now change to spheroidal coordinates (r, μ_i, ϕ_i):

$$(1+\lambda a_i^2)\,y^2\,\hat\mu_i^2 = (r^2 + a_i^2)\,\mu_i^2\,, \qquad \sum_k \mu_k^2 = 1$$

The Kerr-de Sitter Metrics

The de Sitter metric in these coordinates is

$$\begin{aligned} ds^2 &= -W(1-\lambda r^2)dt^2 + F dr^2 \\ &\quad + \sum_i \frac{r^2+a_i^2}{1+\lambda a_i^2}\,(d\mu_i^2 + \mu_i^2\, d\phi_i^2) \\ &\quad + \frac{\lambda}{W(1-\lambda r^2}\left(\sum_i \frac{(r^2+a_i^2)\mu_i d\mu_i}{1+\lambda a_i^2}\right)^2 \end{aligned}$$

where

$$W \equiv \sum_i \frac{\mu_i^2}{1+\lambda a_i^2}, \qquad F \equiv \frac{r^2}{1-\lambda r^2}\sum_i \frac{\mu_i^2}{r^2+a_i^2}$$

The construction of the Kerr-de Sitter metric is now simple. It is given by

$$d\hat{s}^2 = ds^2 + \frac{2M}{U}\,(k_\mu\, dx^\mu)^2$$

where

$$\begin{aligned} k_\mu\, dx^\mu &= W\, dt + F\, dr - \sum_i \frac{a_i\, \mu_i^2\, d\phi_i}{1+\lambda\, a_i^2} \\ U &= \sum_i \frac{\mu_i^2}{r^2+a_i^2}\prod_j (r^2+a_j^2) \end{aligned}$$

The form of these metrics was suggested by generalisation from the known $\lambda = 0$ and $D = 4, 5$ Kerr-de Sitter metrics. Checking that they indeed satisfy the Einstein equations is a mechanical exercise, made (relatively) simple by using the Kerr-Schild evaluation of $\widehat{R}^{\mu}{}_{\nu}$. We have verified this explicitly (using Mathematica) in all dimensions $D \le 11$. Since there are no features of the metrics peculiar to $D \le 11$, we can be confident that the metrics satisfy the Einstein equations for all D.

The metrics in even dimension $D = 2n$ are very similar in structure. Essentially, one again has n latitudinal coordinates μ_i satisfying $\mu_i \, \mu_i = 1$, but now only $(n-1)$ azimuthal coordinates ϕ_i. Correspondingly, there are just $(n-1)$ associated rotation parameters a_i.

Boyer-Lindquist Coordinates

To study the global structure, it is advantageous to change to Boyer-Lindquist type coordinates. After the appropriate coordinate transformations, the metrics become

$$
\begin{aligned}
ds^2 &= -W\,(1-\lambda\,r^2)\,d\tau^2 + \frac{U\,dr^2}{V-2M} \\
&\quad + \frac{2M}{U}(d\tau - \sum_{i=1}^{n} \frac{a_i\,\mu_i^2\,d\varphi_i}{1+\lambda\,a_i^2})^2 \\
&\quad + \sum_{i=1}^{n} \frac{r^2+a_i^2}{1+\lambda\,a_i^2}\,[d\mu_i^2 + \mu_i^2\,(d\varphi_i - \lambda\,a_i\,d\tau)^2] \\
&\quad + \frac{\lambda}{W\,(1-\lambda r^2)}(\sum_{i=1}^{n} \frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1+\lambda\,a_i^2})^2\,,
\end{aligned}
$$

where

$$
\begin{aligned}
V &\equiv \frac{1}{r^2}\,(1-\lambda\,r^2)\,\prod_{i=1}^{n}(r^2+a_i^2) = \frac{U}{F}\,, \\
W &\equiv \sum_i \frac{\mu_i^2}{1+\lambda a_i^2}\,, \qquad F \equiv \frac{r^2}{1-\lambda r^2}\sum_i \frac{\mu_i^2}{r^2+a_i^2}
\end{aligned}
$$

Horizons

Horizons occur where $V(r)$ vanishes; say at $r = r_H$. Introducing angular velocities

$$\Omega^i = \frac{a_i\,(1 + \lambda\, a_i^2)}{r_H^2 + a_i^2}\,,$$

we define the Killing vector

$$\ell = \frac{\partial}{\partial \tau} + \Omega^i\,\frac{\partial}{\partial \varphi^i}$$

This becomes null on the Killing Horizon at $r = r_H$.

The *Surface Gravity* κ is defined by

$$\ell^\nu\,\nabla_\nu\,\ell_\mu = \kappa\,\ell_\mu$$

restricted to the horizon, leading to

$$\kappa = r_H\,(1 - \lambda\, r_H^2)\,\sum_i \frac{1}{r_H^2 + a_i^2} - \frac{1}{r_H}$$

From this, one obtains the *Hawking Temperature*

$$T_H = \frac{\kappa}{2\pi}$$

Complete Euclidean-Signature Einstein Metrics

Real metrics of positive-definite metric signature can be obtained by sending

$$\tau \longrightarrow -i\,\tau\,, \qquad a_i \longrightarrow i\,a_i$$

In 1978, Page obtained the first example of a complete, non-singular Einstein metric on a compact manifold (S^2 bundle over S^2), by taking a limit of the Euclideanised four-dimensional Kerr-de Sitter metric. This was generalised to five dimensions in 2004, by Hashimoto, Sakaguchi and Yasui. We can now extend this to arbitrary dimensions.

Compact metric achieved when radial coordinate ranges between two zeros $r = r_1$ and $r = r_2$ of $V(r)$. Metric singularities at the endpoints $\mu_i = 0$ of the latitutinal coordinates μ_i are removable if the azimuthal angles φ^i have period 2π.

Near endpoints $r = r_1$ and $r = r_2$, the metric takes the form (e.g. $r = r_1$):

$$\begin{aligned} ds^2 \;\approx\; & d\rho^2 + \kappa_1^2 \rho^2 d\tau^2 + g_{ij}(d\varphi^i - \Omega_1^i d\tau)(d\varphi^j - \Omega_1^j d\tau) \\ & + \bar{g}_{ij} d\mu^i d\mu^j \end{aligned}$$

where κ_1 is the surface gravity and Ω_1^i is the angular velocity at $r = r_1$. The conical singularity at $\rho = 0$ is removable if τ has period $2\pi/\kappa_1$ at fixed

$$\varphi_1^i \equiv \varphi^i - \Omega_1^i \tau$$

Define a new Euclidean "time" coordinate $\psi_1 \equiv \kappa_1 \tau$ that has period 2π.

We need local charts with coordinates (ψ_1, φ_1^i) and (ψ_2, φ_2^i) at $r = r_1$ and $r = r_2$. All periods are 2π. Transition functions are given by the matrix S of linear transformations

$$\begin{pmatrix} \psi_2 \\ \varphi_2^i \end{pmatrix} = \begin{pmatrix} \frac{\kappa_2}{\kappa_1} & 0 \\ \frac{\Omega_1^i - \Omega_2^i}{\kappa_1} & \delta_j^i \end{pmatrix} \begin{pmatrix} \psi_1 \\ \varphi_1^j \end{pmatrix}$$

The transition matrix S must be invertible, and both S and S^{-1} must have integer entries, since all coordinate periods are 2π. Thus S must be an $SL(n+1,Z)$ matrix, implying

$$|\kappa_1| = |\kappa_2| \equiv \kappa\,, \qquad \Omega_1^i - \Omega_2^i = \kappa\, k_i$$

for integers k_i.

The construction then gives T^n bundles over S^2, characterised by the winding numbers k_i of the image of the equator around the n cycles of T^n. (The S^2 is coordinatised by (r,ψ).) The metrics extend smoothly onto manifolds that are associated S^{D-2} bundles over S^2 with structure group T^n (acting on the azimuthal coordinates φ^i).

The $SL(n+1,Z)$ conditions are satisfied by choosing the parameters M and a_i so that $r_1 = r_2$ (and rescaling to a non-singular radial coordinate covering this non-zero proper-distance interval), and, with the limit $r_2 \to r_1$ (and hence $\kappa \to 0$),

$$\frac{a_i\,(1-\lambda\,a_i^2)(r_2^2-r_1^2)}{\kappa\,(r_1^2-a_i^2)(r_2^2-a_i^2)} = k_i$$

Charged Kerr-de Sitter Black Holes

In four dimensions, charged Kerr-de Sitter black holes are well known. They are solutions of the coupled Einstein-Maxwell equations with a cosmological constant. (Kerr-Newman-de Sitter; found by Carter, 1968.)

It is not immediately obvious what is the natural generalisation to consider in higher dimensions: For example, pure Einstein-Maxwell solutions, or Einstein-Maxwell with additional "Chern-Simons" terms? (The latter is dimension-specific.)

From a string theory viewpoint, it is most natural to consider charged black-hole solutions in supergravity theories; this allows the possibility of supersymmetric (BPS) special cases.

Since this becomes a dimension-specific question, let us focus on $D = 5$.

Charged Rotating Black Holes in $D = 5$ Gauged Supergravity

M. Cvetič, H. Lü and C.N. Pope

The Lagrangian is

$$\mathcal{L} = \sqrt{-g}(R - 12\lambda - \tfrac{1}{4}F^2 + \tfrac{1}{12\sqrt{3}}\,\epsilon^{\mu\nu\rho\sigma\lambda}\,F_{\mu\nu}\,F_{\rho\sigma}\,A_\lambda)$$

In five dimensions, there can be two independent rotation parameters. Finding the general such charged solution is rather complicated. We shall specialise to the case where the two rotation parameters are set equal.

A useful guide to the form of the solution is that it should specialise to results of Hawking, Hunter and Taylor-Robinson when the charge is zero. It should also specialise to results of Cvetič and Youm when the cosmological constant is zero. With these in mind, we made an ansatz, and fixed functional dependences by imposing the equations of motion. This leads to:

The Charged Solution

$$
\begin{aligned}
ds_5^2 = & -(1 - \Sigma\,\lambda r^2 - \frac{2M}{r^2} + \frac{Q^2}{r^4})\,dt^2 + \frac{dr^2}{W} + r^2\,d\Omega_3^2 \\
& -\frac{J^2}{r^4}\,[Q^2 - 2(M+Q)\,r^2]\,(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi)^2 \\
& -2J\,(\lambda\,\beta\,r^2 + \frac{2M+Q}{r^2} - \frac{Q^2}{r^4}) \times \\
& \quad dt\,(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi)
\end{aligned}
$$

$$
A = \frac{\sqrt{3}Q}{r^2}\,[dt - J\,(\sin^2\theta\,d\phi + \cos^2\theta\,d\psi)]
$$

where

$$
\begin{aligned}
W &= 1 - \lambda\,r^2 + \frac{c_1}{r^2} + \frac{c_2}{r^4} \\
\Sigma &= 1 + \lambda\,\beta^2\,J^2 \\
c_1 &\equiv 2M + 2\lambda\,J^2\,(M+Q) - 2\lambda\,J^2(2M+Q)\,\beta \\
& \quad +2\lambda^2\,J^4\,(M+Q)\,\beta^2 \\
c_2 &\equiv (\lambda\,\beta\,J^2 - 1)^2\,Q^2 + J^2\,(\lambda\,Q^2 + 2(M+Q))
\end{aligned}
$$

and

$$
d\Omega_3^2 = d\theta^2 + \sin^2\theta\,d\phi^2 + \cos^2\theta\,d\psi^2
$$

is the metric on the unit 3-sphere.

There are four non-trivial parameters: (M, J, Q, β). The first three correspond to mass, rotation and charge. The meaning of β, which is trivial (absorbable by rescaling) when $\lambda = 0$, but non-trivial when $\lambda \neq 0$, is not so clear.

A simpler presentation of the metric, in terms of left-invariant 1-forms σ_i on S^3, is

$$ds^2 = -\frac{r^2 W\, dt^2}{4b^2} + \frac{dr^2}{W} + \frac{r^2}{4}(\sigma_1^2 + \sigma_2^2) + b^2(\sigma_3 + f\, dt)^2$$

where

$$\begin{aligned} b^2 &= \tfrac{1}{4} r^2 \left(1 - \frac{J^2 Q^2}{r^6} + \frac{2J^2 (M+Q)}{r^4}\right) \\ f &= -\frac{J}{2b^2}\left(\lambda \beta r^2 + \frac{2M+Q}{r^2} - \frac{Q^2}{r^4}\right) \end{aligned}$$

One can consider "extremal" cases, where $Q = \pm M$. The solutions with $Q = +M$ include the BPS solutions found by Gutowski and Reall. The solutions with $Q = -M$ include BPS solutions found by Klemm and Sabra:

$$\text{G-R}: \quad Q = M\,, \quad J = \tfrac{1}{2}\sqrt{-\lambda}\, M\,, \quad \beta = -\frac{2}{\lambda M}$$

$$\text{K-S}: \quad Q = -M\,, \quad \beta = 0$$

Global Structure of the Metrics

More generally, all $Q = -M$ solutions (arbitrary β) are BPS. Also all $Q = +M$ solutions with $\beta^2 = -1/(\lambda J^2)$.

The metric has event horizons where $W(r) = 0$. Suppose $\lambda < 0$ (AdS), and let $r = r_+$ be the outermost horizon. To avoid the curvature singularity at $r = 0$, we must have $r_+ > 0$. There will be naked *Closed Timelike Curves* (CTCs) unless $b(r)^2 > 0$ for all $r \geq r_+$.

The Klemm-Sabra solution, i.e. $Q = -M$, $\beta = 0$, has CTCs. The more general solutions obtained by relaxing the $\beta = 0$ condition with $Q = -M$ again have CTCs.

The Gutowski-Reall solution has no CTCs. The more general $Q = +M$ supersymmetric solutions have CTCs.

The general non-supersymmetric solutions, with the four free parameters (M, J, Q, β), will be regular (i.e. no singularities or CTCs outside the horizon) for appropriate ranges of the parameters.