

Non-Relativistic

Gauge-Gravity Dualities

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Based on

Galilean Conformal Algebra:

- ❖ Arjun Bagchi and Rajesh Gopakumar, (arXiv: 0902.1385).
- ❖ Arjun Bagchi, Ipsita Mandal (arXiv: 0903.4524).
- ❖ M. Alishahiha et.al. (arXiv: 0903.3953), D. Martelli and Y. Tachikawa (arXiv: 0903.5184)
C. Duval and P. Horvathy (arXiv: 0904.1531).

Schrodinger Algebra:

- ❖ Y. Nishida and D. T. Son, Phys. Rev. D **76**, 086004 (2007) and D. T. Son Phys. Rev. D **78**, 046003 (2008).
- ❖ K. Balasubramaniam and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008).

Outline of the talks

◆ Gauge-Gravity Dualities

- ✧ The AdS/CFT Conjecture

◆ Two Kinds of Non-Relativistic Limits

- ✧ Schrodinger Symmetry; Galilean Conformal Symmetry

◆ Galilean Conformal Symmetry

- ✧ Virasoro-Kac-Moody Symmetry

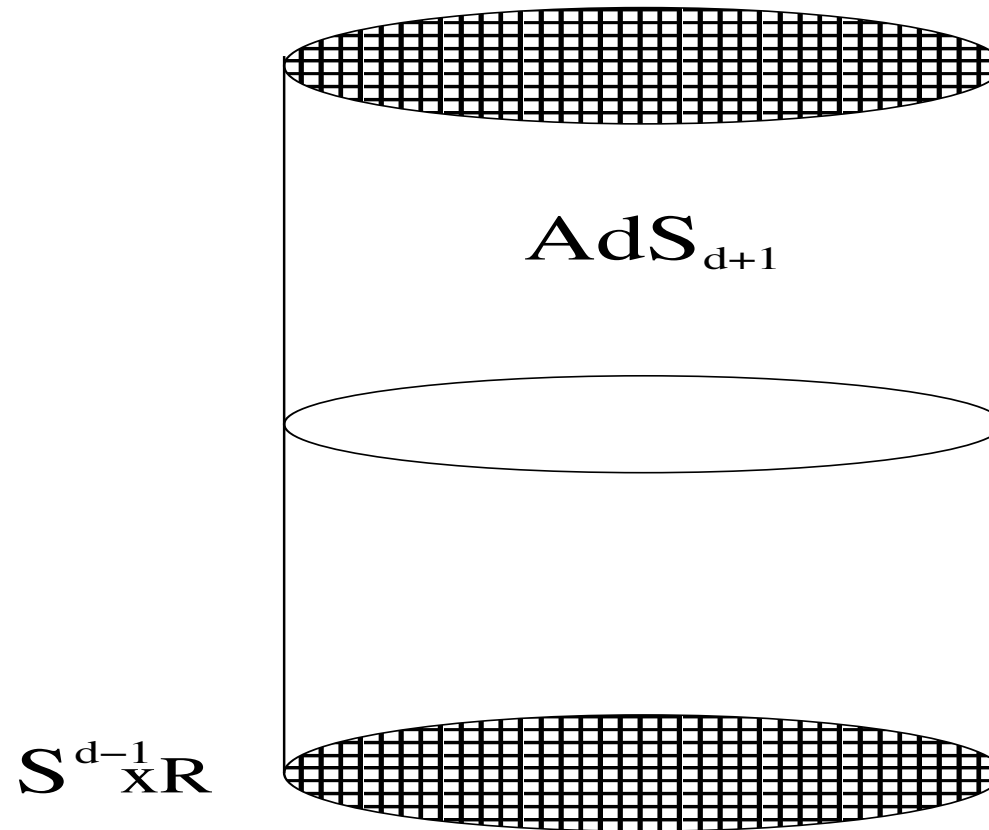
◆ Gravity Duals

- ✧ Spacetime Geometries for Schrodinger/Galilean Conformal Symmetries

1 Gauge-Gravity Dualities

Gauge-Gravity Dualities

- ◆ Gauge-Gravity dualities offer a radically new perspective into the dynamics of (some) **strongly interacting relativistic** gauge theories.



Equivalent description in terms of *gravitational* physics in one (or more) higher dimension.

Gauge-Gravity Dualities *continued*

SALIENT FEATURES

- ◆ $3 + 1$ dimensional Quantum Field Theory living on the *boundary* of AdS_5 spacetime – Holography.
- ◆ Strongly interacting field theory \leftrightarrow Weakly curved gravity (Einstein's Equations).
- ◆ Correlation functions of four dimensional field theory \leftrightarrow Semi-classical action of Gravity (with sources).
- ◆ Symmetries of field theory \leftrightarrow Isometries of the Spacetime Geometry
- ◆ Finite temperature field theory \leftrightarrow “Thermal Quantum Gravity” (e.g. physics of black holes).
- ◆ Leads to predictions for strong coupling values of transport coefficients in a number of field theories.

Gauge-Gravity Dualities *continued*

SALIENT FEATURES *continued*

- ◆ **Prototypical example:** Maximally Supersymmetric ($\mathcal{N} = 4$) Yang-Mills Theory dual to IIB string theory (supergravity) on $AdS_5 \times S^5$.
- ◆ A **conformally invariant** interacting field theory (CFT).
- ◆ Has successfully passed many, many non-trivial tests.
- ◆ A line of fixed points parametrised by a coupling λ .
- ◆ $SO(4, 2)$ relativistic conformal group \leftrightarrow $SO(4, 2)$ Isometry group of AdS_5 .
- ◆ $SO(4, 2)$ contains Poincare group $ISO(3, 1)$ plus dilatations D and special conformal transformations K_μ .
- ◆ By taking parametric limits of the parent conjecture one can focus on special kinematic subsectors (e.g. the **BMN limit**).

2 Taking Non-Relativistic Limits of the Gauge-Gravity Duality

Non-Relativistic Limits

- ◆ Useful limit which might help understand a larger class of systems (real world?).
- ◆ Two kinds of nonrelativistic limits of relativistic systems.
- ◆ For massive systems, can consider the limit where the rest energy \gg kinetic energy.
- ◆ Replace $\partial_0 \rightarrow -im_0 + \partial_t$; $m_0 \rightarrow \frac{m}{\epsilon^2}$; $x_i \rightarrow \epsilon x_i$ (with $\epsilon \rightarrow 0$).
- ◆ Then Klein Gordon equation reduces to Schrodinger equation

$$(\partial_0^2 - \partial_i^2 + m_0^2)\phi = 0 \rightarrow (i\partial_t + \frac{1}{2m}\partial_i^2)\phi = 0.$$

- ◆ The parameter $\epsilon \sim \frac{v}{c} \rightarrow 0$ signifies taking the nonrelativistic limit.
- ◆ What are the symmetries in this limit?

Schrodinger Symmetry

- ◆ The set of symmetries of the free particle Schrodinger equation, $(i\partial_t + \frac{1}{2m}\partial_i^2)\phi = 0$ is called the **Schrodinger** Symmetry $Sch(3, 1)$.
- ◆ It is sometimes referred to as a non-relativistic analogue of conformal symmetry.
- ◆ This symmetry also believed to arise in interacting systems like those of fermions whose scattering length becomes infinite.
- ◆ Realised in cold atom systems with coupling tuned between BEC and BCS transition.
- ◆ $Sch(3, 1)$ contains all the usual **Galilean** symmetries $G(3, 1)$:

$$\begin{aligned} [J_{ij}, J_{rs}] &= so(3) \\ [J_{ij}, B_r] &= -(B_i\delta_{jr} - B_j\delta_{ir}) \\ [J_{ij}, P_r] &= -(P_i\delta_{jr} - P_j\delta_{ir}), \quad [J_{ij}, H] = 0 \\ [B_i, B_j] &= 0, \quad [P_i, P_j] = 0, \quad [B_i, P_j] = m\delta_{ij} \\ [H, P_i] &= 0, \quad [H, B_i] = -P_i. \end{aligned} \tag{1}$$

Schrodinger Symmetry *continued*

- ✧ But also has *two* more generators \tilde{K}, \tilde{D}
- ✧ \tilde{D} is a dilatation operator which scales time and space *differently*

$$x_i \rightarrow \lambda x_i, \quad t \rightarrow \lambda^2 t. \quad (2)$$

- ✧ \tilde{K} is like a time component of special conformal transformations.

$$x_i \rightarrow \frac{x_i}{(1 + \mu t)}, \quad t \rightarrow \frac{t}{(1 + \mu t)}. \quad (3)$$

- ✧ No analogue of the spatial components K_i of special conformal transformations.
- ✧ Thus smaller group compared to the relativistic conformal group: 12 parameters (+ central mass term) as opposed to 15 parameters for $SO(4, 2)$.

Galilean Conformal Symmetry

- ✧ Second kind of non-relativistic limit appropriate for (massless) conformal field theories.
- ✧ Here the starting symmetry group is bigger e.g. $SO(4, 2)$.
- ✧ Taking the non-relativistic limit now means taking a *group contraction* of this group.
- ✧ Generalisation of the process by which one recovers the **Galilean group** $G(d, 1)$ from the **Poincare group** $ISO(d, 1)$.
- ✧ Take $t \rightarrow t$ and $x_i \rightarrow \epsilon x_i$ and scale $\epsilon \rightarrow 0$.
- ✧ Thus $v_i \sim \epsilon \Rightarrow$ non-relativistic limit.
- ✧ Poincare generators reduce to the **Galilean generators** (after appropriate rescaling)

$$\begin{aligned} H &= -\partial_t, & P_i &= \partial_i \\ B_i &= t\partial_i, & M_{ij} &= -(x_i\partial_j - x_j\partial_i). \end{aligned} \tag{4}$$

Galilean Conformal symmetry *continued*

- ◆ Now extend this scaling to all the extra generators of the relativistic conformal group $SO(4, 2)$ i.e (D, K_0, K_i) (Lukierski et.al.; Gomis, Gomis and Kamimura)
- ◆ Gives the contracted vector fields

$$\begin{aligned} D &= -(x_i \partial_i + t \partial_t) \\ K &= -(2tx_i \partial_i + t^2 \partial_t) \\ K_i &= t^2 \partial_i \end{aligned} \tag{5}$$

- ◆ Note the dilatation generator D is the *same* as in the relativistic theory. $x_i \rightarrow \lambda x_i, t \rightarrow \lambda t$ ($z = 1$ scaling: See Henkel).
- ◆ Therefore different from $\tilde{D} = -(2t\partial_t + x_i \partial_i)$.
- ◆ Spatial conformal transformation generators K_i are present and generate constant acceleration transformations. $x_i \rightarrow x_i + \frac{1}{2}a_i t^2$.
- ◆ The temporal special conformal generator different from $\tilde{K} = -(tx_i \partial_i + t^2 \partial_t)$

Galilean Conformal Symmetry *continued*

- ❖ The algebra of these generators (together with that of the Galilean Algebra) is quite **different** from the Schrodinger group.
- ❖ Note we now have **15** generators as opposed to **12** in $Sch(3, 1)$.
- ❖ Galilean central mass term in $[B_i, P_j]$ not admissible here – "**massless non-relativistic system**".
- ❖ The Galilean conformal symmetries are actually realised on the Euler equations of fluid mechanics. (With K acting trivially) (Bhattacharya, Minwalla and Wadia).

$$\partial_t v_i(x_i, t) + v_j \partial_j v_i(x_i, t) = -\partial_i p(x_i, t). \quad (6)$$

- ❖ In fact, these equations admit a much larger symmetry. Under arbitrary boosts $x_i \rightarrow x_i + b_i(t)$.

Galilean Conformal Symmetry *continued*

❖ Algebra of the contracted conformal group: Define

$$\begin{aligned} L^{(-1)} &= H, & L^{(0)} &= D, & L^{(+1)} &= K, \\ M_i^{(-1)} &= P_i, & M_i^{(0)} &= B_i, & M_i^{(+1)} &= K_i. \end{aligned} \quad (7)$$

❖ Then

$$\begin{aligned} [J_{ij}, L^{(n)}] &= 0, & [L^{(m)}, M_i^{(n)}] &= (m - n)M_i^{(m+n)} \\ [J_{ij}, M_k^{(m)}] &= -(M_i^{(m)}\delta_{jk} - M_j^{(m)}\delta_{ik}), & [M_i^{(m)}, M_j^{(n)}] &= 0, \\ [L^{(m)}, L^{(n)}] &= (m - n)L^{(m+n)}. \end{aligned} \quad (8)$$

❖ Note the $SL(2, R)$ algebra in the last line. *Different* from that in the Schrodinger group.

Non-Relativistic Conformal Symmetries - Correlation Functions

- ❖ We can use the Schrodinger/Galilean Conformal symmetry to constrain two and three point functions.
- ❖ In the Schrodinger case, define Quasi-Primary Operators which obey $[B_i, \mathcal{O}] = [\tilde{K}, \mathcal{O}] = 0$.
- ❖ These primary operators are labelled by their eigenvalue under \tilde{D} and m .

$$G_{12}(\Delta x_i, \Delta t) = C_{12} \delta_{h_1, h_2} \delta_{m_1, m_2} (\Delta t)^{-h} \exp \frac{m(\Delta x_i)^2}{2\Delta t}.$$

$$G_{123}(\vec{x}^{(a)}, t^{(a)}) = C_{123} \delta_{m_1+m_2+m_3, 0} (t_{12})^{\frac{h_3-h_1-h_2}{2}} (t_{23})^{\frac{h_1-h_2-h_3}{2}} (t_{31})^{\frac{h_2-h_3-h_1}{2}} \exp\left(\frac{m_1(x_{13})^2}{2t_{13}} + \frac{m_2(x_{23})^2}{2t_{23}}\right) f\left(\frac{[x_{13}t_{23} - x_{23}t_{13}]^2}{t_{12}t_{23}t_{31}}\right). \quad (9)$$

Non-Relativistic Conformal Symmetries - Correlation

Functions *continued*

- ❖ In the GCA case, the situation is closely parallel to the relativistic conformal case. Two and three point functions are essentially fixed.
- ❖ Define Quasi-Primary Operators which obey $[K_i, \mathcal{O}] = [K, \mathcal{O}] = 0$.
- ❖ These primary operators are labelled by their eigenvalues (h, ξ_i) under $D = L^{(0)}$ and $B_i = M_0^{(i)}$.

$$G_{12}(\Delta x_i, \Delta t) = C_{12} \delta_{h_1, h_2} \delta_{\xi_1 + \xi_2, 0} (\Delta t)^{-2h} \exp\left(\frac{2\xi^i \Delta x_i}{\Delta t}\right).$$

$$G_{123}(\vec{x}^{(a)}, t^{(a)}) = C_{123} \delta_{\xi_1^i + \xi_2^i + \xi_3^i, 0} (t_{12})^{h_3 - h_1 - h_2} (t_{23})^{h_1 - h_2 - h_3} (t_{31})^{h_2 - h_3 - h_1} \exp\left(\frac{2\xi_1^i x_{23}^i}{t_{23}} + \frac{2\xi_2^i x_{31}^i}{t_{31}} + \frac{2\xi_3^i x_{12}^i}{t_{12}}\right). \quad (10)$$

3 Extended Galilean Conformal Symmetry

Extended Galilean Conformal Symmetry

- ❖ A remarkable feature of this algebra is that it admits a very natural extension to an infinite dimensional $SO(d)$ Current Algebra.
- ❖ Define the vector fields for arbitrary integer n

$$\begin{aligned}L^{(n)} &= -(n+1)t^n x_i \partial_i - t^{n+1} \partial_t \\M_i^{(n)} &= t^{n+1} \partial_i \\J_{ij}^{(n)} &= -t^n (x_i \partial_j - x_j \partial_i)\end{aligned}\tag{11}$$

- ❖ They obey exactly the same commutation relations as the ones for $m, n = 0, \pm 1$.

$$\begin{aligned}[L^{(m)}, L^{(n)}] &= (m-n)L^{(m+n)} & [L^{(m)}, J_a^{(n)}] &= nJ_a^{(m+n)} \\[J_a^{(n)}, J_b^{(m)}] &= f_{abc}J_c^{(n+m)} & [L^{(m)}, M_i^{(n)}] &= (m-n)M_i^{(m+n)}\end{aligned}\tag{12}$$

- ❖ The Virasoro and Kac-Moody algebra of the vector fields is, of course, without the central extension.

Extended Galilean Conformal Symmetry *continued*

- ❖ The $M_i^{(n)}$ act as generators of generalised time dependent but spatially homogeneous accelerations

$$x_i \rightarrow x_i + b_i(t). \quad (13)$$

- ❖ This is same symmetry possessed by the Euler equations.
- ❖ Similarly, the $J_{ij}^{(n)} \equiv J_a^{(n)}$ are generators of arbitrary time dependent rotations

$$x_i \rightarrow R_{ij}(t)x_j \quad (14)$$

- ❖ These two together generate what is sometimes called the **Coriolis group**: the biggest group of "isometries" of "flat" Galilean spacetime.
- ❖ $L^{(n)}$ seem to be generators of a **conformal "isometry"** of Galilean spacetime.

$$t \rightarrow f(t), \quad x_i \rightarrow \frac{df}{dt}x_i \quad (15)$$

- ❖ All these generators together describe, in fact, the natural set of conformal isometries of Galilean spacetime.

4 Gravity Duals

Gravity Duals

- ❖ What can we say about the spacetime geometries dual to a system with non-relativistic Conformal Symmetry (either Schrodinger or Galilean Conformal)?
- ❖ We would like to have a spacetime which has these symmetries as isometries (in an appropriate sense).
- ❖ The corresponding spacetimes are somewhat unfamiliar.
- ❖ In the case of the Schrodinger Symmetry, it is a spacetime in **six** dimensions, with an identification along a null direction (Dario's talk).



$$ds^2 = \frac{2dx^+ dx^- - dx^i dx^i - dz^2}{z^2} + 2\frac{(dx^+)^2}{z^4}. \quad (16)$$

- ❖ The first piece is the metric of AdS_6 while the second is a deformation.
- ❖ The null direction x^- is identified with a period proportional to the mass (or particle number density).

The Gravity Dual *continued*

- ❖ In the case of the Galilean Conformal Algebra, the spacetime is five dimensional, obtained by taking a Galilean scaling limit of the AdS_5 geometry.
- ❖ However, the metric degenerates on taking the Galilean limit – spatial interval scales to zero.
- ❖ There is a surviving AdS_2 piece and a degenerate Euclidean metric on the remaining R^3 directions.
- ❖ Might look like a singular limit. However, analogous to taking a Newtonian limit of Einstein's equations.
- ❖ Well-defined geometric theory of newtonian gravitation - **Newton-Cartan theory**.
- ❖ Spacetime with a non-metric dynamical connection.

In our case, can be viewed as connection of an R^3 fibre bundle over AdS_2 .

- ❖ Has the right asymptotic symmetries – the infinite dimensional extension of the Galilean conformal algebra.

The Bulk Dual *Symmetries*

❖ AdS_{d+2} in Poincare coordinates:

$$ds^2 = R^2 \frac{dt^2 - dz^2 - dx_i^2}{z^2} \quad (17)$$

❖ In radially infalling coordinates for null geodesics ($t' = t + z, z' = z$)

$$ds^2 = \frac{R^2}{z'^2} (-2dt' dz' + dt'^2 - dx_i^2) = \frac{R^2}{z'^2} (-dt'(2dz' - dt') - dx_i^2). \quad (18)$$

❖ Take the generators of the AdS_{d+2} isometries and perform the contraction by taking $t', z' \rightarrow \epsilon^r, x_i \rightarrow \epsilon^{r+1} x_i$. Metric degenerates as expected.

❖ Contracted Killing vectors given by

$$\begin{aligned} P_i &= -\partial_i, & B_i &= -(t' - z')\partial_i, & K_i &= -(t'^2 - 2t'z')\partial_i \\ H &= \partial_{t'}, & D &= t'\partial_{t'} + z'\partial_{z'} + x_i\partial_i, & K &= t'^2\partial_{t'} + 2(t' - z')(z'\partial_{z'} + x_i\partial_i). \end{aligned} \quad (19)$$

The Bulk Dual *Symmetries*

★ More compactly (for $m, n = 0, \pm 1, l = 0$).

$$\begin{aligned} L^{(n)} &= t'^{n+1} \partial_{t'} + (n+1)(t'^n - nzt'^{n-1})(x_i \partial_i + z' \partial_{z'}) \\ M_i^{(m)} &= -(t'^{m+1} - (m+1)zt'^m) \partial_i \\ J_{ij}^{(l)} &= -t'^n (x_i \partial_j - x_j \partial_i) \end{aligned} \tag{20}$$

- ★ Reduces at the boundary ($z = 0$) to the generators of the contracted conformal algebra. *And satisfies the same algebra.*
- ★ *In fact, these bulk vector fields (for arbitrary m, n, l) reduce to that of the extended Kac-Moody algebra at the boundary.*
- ★ *What is the role of these vector fields in the bulk?*
- ★ The *Virasoro* generators act as the generators of asymptotic symmetries of the *AdS₂*.
- ★ *The others act only on the R^3 (like Galilean "isometries" on the boundary).*

To Summarise

Summary

- ☆ Gauge-Gravity dualities can be generalised to a non-relativistic setting.
- ☆ Galilean conformal symmetry relevant to "massless" non-relativistic systems.
- ☆ Identify the sector in e.g. $\mathcal{N} = 4$ SYM described by the GCA.
- ☆ Do the ward identities of the full GCA constrain correlators in this sector more than expected otherwise?
- ☆ Are there real life systems which are described by the GCA?
- ☆ Need to develop a better understanding of the gravity duals which involve novel features such as the Newtonian limit.
- ☆ Spell out the bulk-boundary dictionary, as a parametric limit of the relativistic case.

Obrigado