NON-PERTURBATIVE ASPECTS OF THE TOPOLOGICAL STRING

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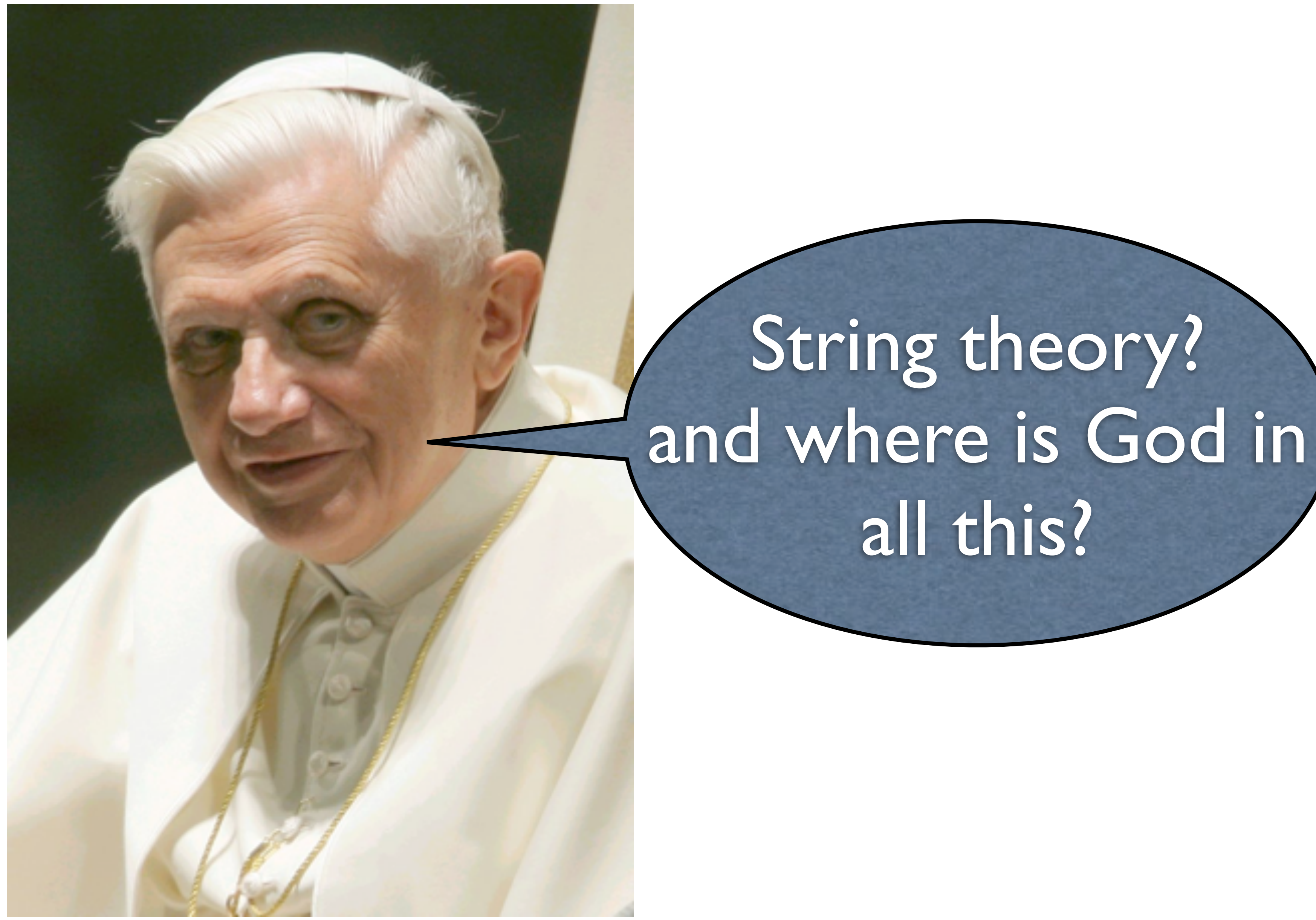
[M.M.-Schiappa-Weiss, 0711.1954 & 0809.2619]
[M.M., 0805.3033]
[B. Eynard-M.M., 0810.4273]
related: [S. Garoufalidis-M.M., 0811.1195]

A CONSERVATIVE POINT OF VIEW 1



Goals

- Improve our understanding of nonperturbative effects in topological string theory (w.r.t. the string coupling constant) and in the $1/N$ expansion

- *New mathematical structures and new information about the old ones:* D-instanton sector of topological string theory.

- Mathematical bonus: *asymptotic properties* at large genus of perturbative amplitudes (with applications in combinatorics, enumerative geometry...).

- Basics of topological strings. A and B models. CYs
- Nonperturbative structure as a function of CY moduli. Phases. Modular properties of the amplitudes
- Nonperturbative effects in the string coupling constant: motivation and perturbative/nonpertubative connection
- Review of matrix models. Large N duals of topological strings
- Instantons from large N duals: the case of 2d gravity
- Instantons in the full matrix model and applications to topological strings
- A holomorphic and background independent nonperturbative partition function: a proposal
- Conclusions and open problems

What is topological string theory?

Topological string theory can be regarded as a toy model of a full-fledged string theory. In terms of *complexity:*

Noncritical strings < Topological strings < Superstrings

The starting point is a *2d topological field theory*, the topological sigma model, with action

$$S = \frac{1}{\ell_s^2} \int \mathrm{d}z \sqrt{g}\, g^{\mu\nu}\, G_{I\overline{J}} \partial_\mu \phi^I \partial_\nu \phi^{\overline{J}} + \cdots$$

The target of this sigma model will be taken to be a Calabi-Yau threefold. We recall that this is a six-dimensional, Kahler manifold which satisfies Einstein's equations in the vacuum

$$R_{IJ} = 0$$

The A and the B models

As in usual string theory, we can study the target metric by perturbing the 2d action with graviton vertex operators. However, due to the topological nature of the theory, only a limited set of fluctuations can be incorporated. For a CY

$$\mathcal{M}_{\rm metrics} = \mathcal{M}_{\rm Kahler} \times \mathcal{M}_{\rm complex}$$

There are *two* versions of topological string theory: the A model incorporates the Kahler parameters ("sizes"), while the B model incorporates complex parameters ("shapes").
We will then study the free energies at genus g as a function of these parameters

$$F_g^A(t_{\rm Kahler}) \qquad F_g^B(t_{\rm complex})$$

Mirror symmetry

A fundamental duality in string theory: given a CY manifold X, there is generically another (topologically different) CY manifold $\widetilde{X}$ such that

$$\mathcal{M}_{\text{Kahler}}(X) = \mathcal{M}_{\text{complex}}(\widetilde{X})$$

$$F_g^A(X; t_{\text{Kahler}}) = F_g^B(\widetilde{X}; t_{\text{complex}})$$

For the last equality to make sense, we must have a map

$$t_{\text{Kahler}} \leftrightarrow t_{\text{complex}}$$

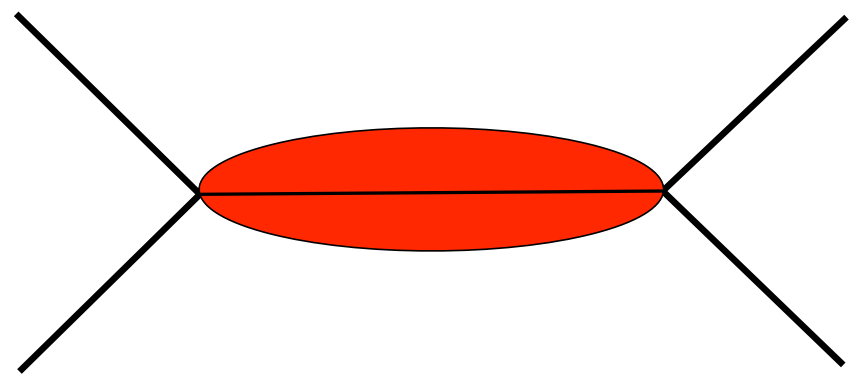
This is called the *mirror map*

Examples I

A very useful example of CY backgrounds are the so-called *toric CYs*. These are *non-compact*, but they have large N duals, as we will discuss

Simplest example: the *resolved conifold*. The moduli space of Kahler deformations is just $t \in \mathbb{C}$, which describes the complexified size of the two-sphere

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{S}^2$$



mirror:

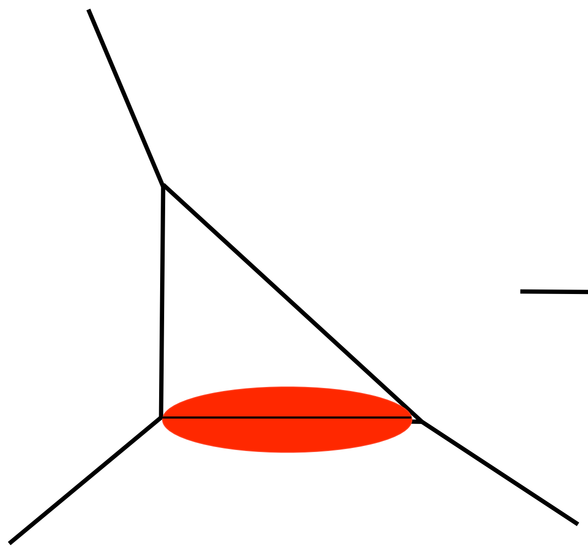
$$\widetilde{X}: \quad uv = P(x,y)$$

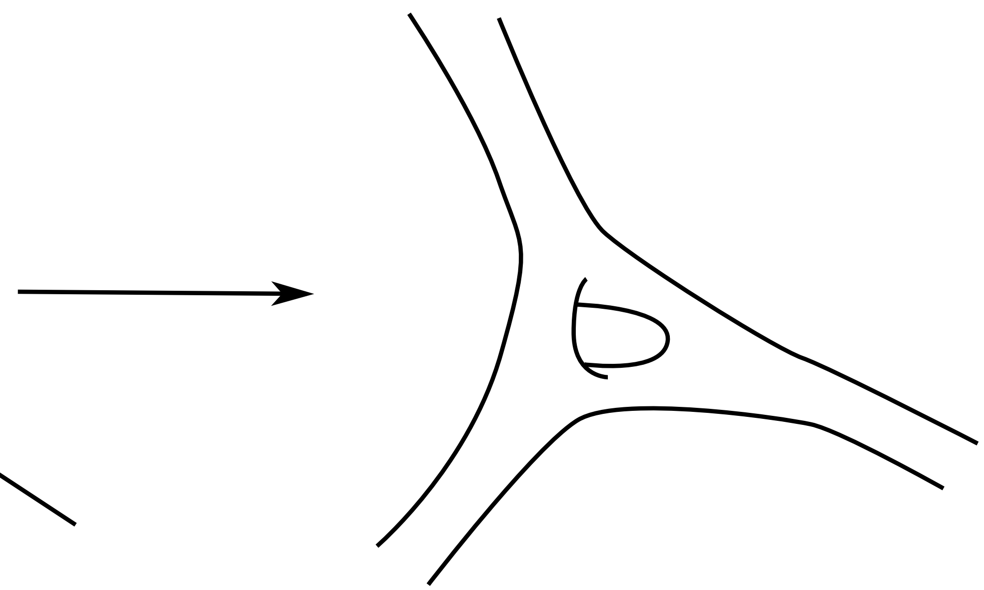
$$P(x,y) = 1 + x + y + \mathrm{e}^{-t} xy$$

⟵ spectral curve

Examples 2

local $\mathbb{P}^2$





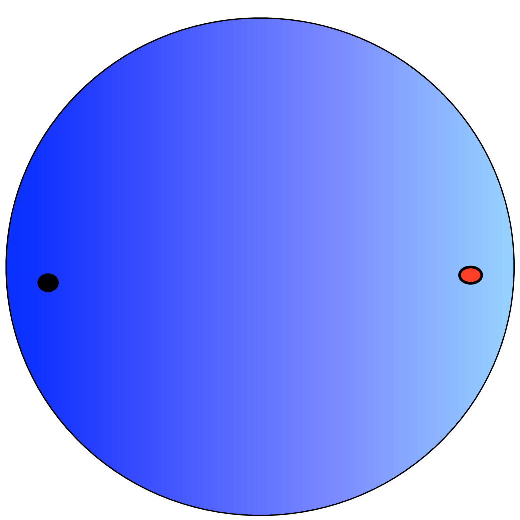
$X = \mathcal{O}(-3) \to \mathbb{P}^2$

$\widetilde{X}: \quad uv = P(x,y)$

$P(x,y) = y(y+x+1) + z_t x^3$

Moduli space

$t = \infty$



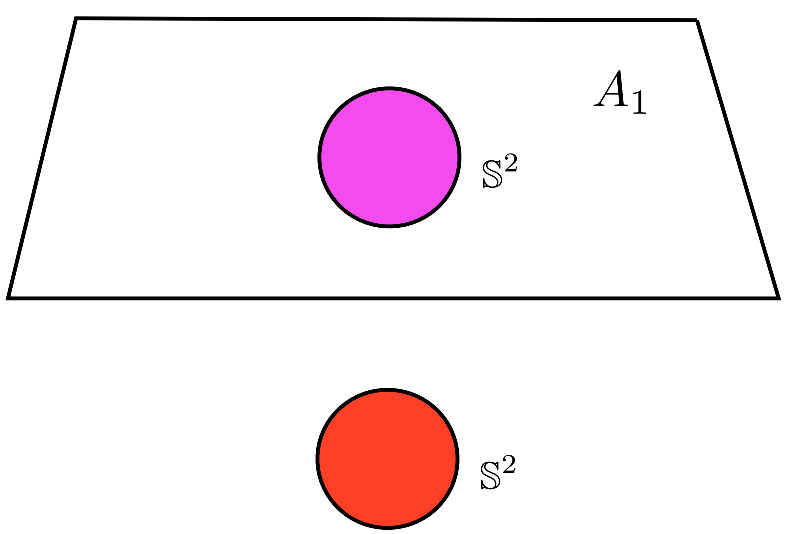
$t = 0 \ \to \mathbf{C}^3/\mathbf{Z}_3$

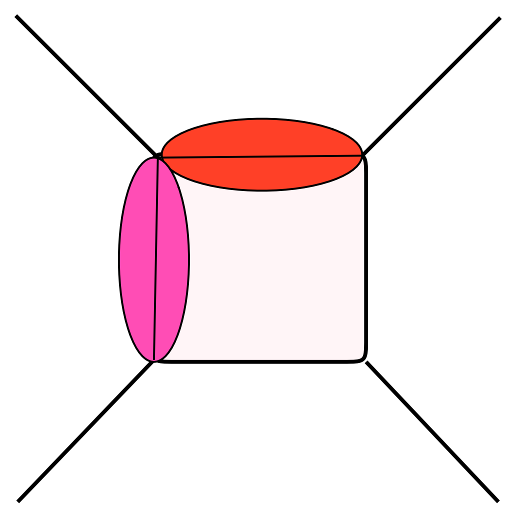
large radius

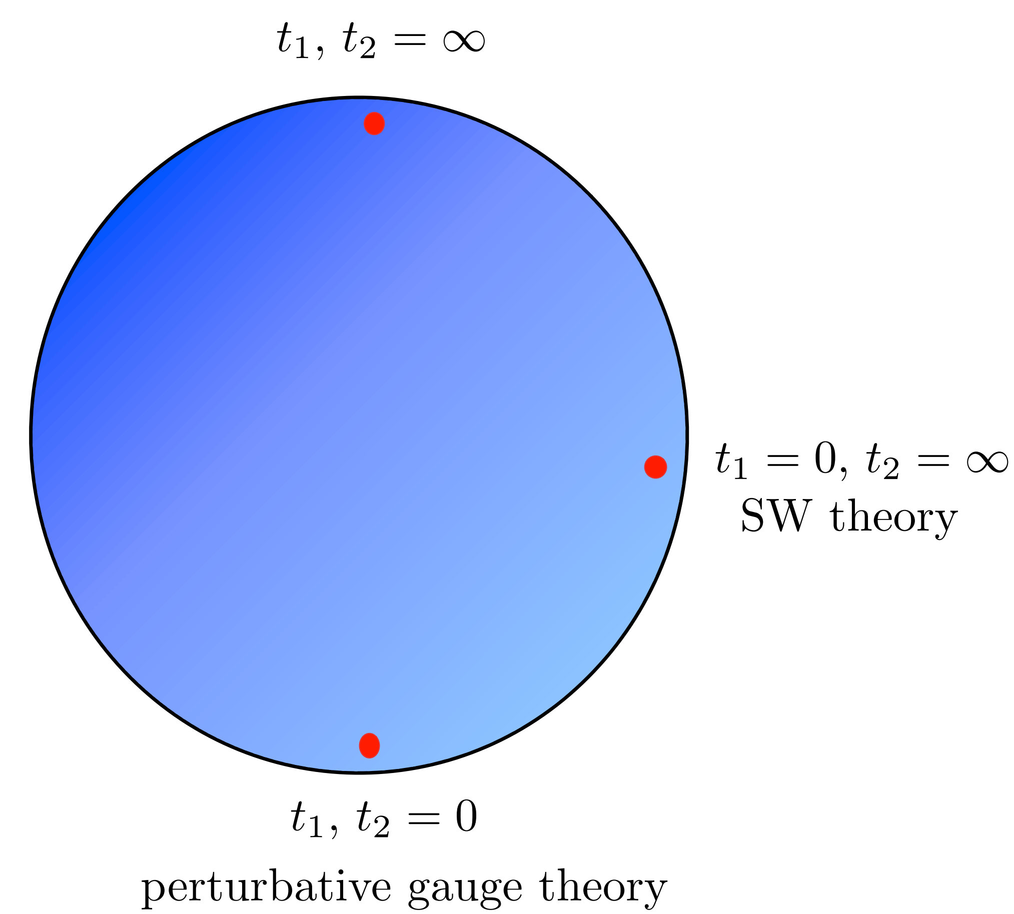
orbifold point

Examples 3

local $\mathbb{P}^1 \times \mathbb{P}^1$: fibration of the A_1 4d singularity over $\mathbb{P}^1$.
It has two Kahler deformations t_1, t_2 (size of the base + $\mathbb{P}^1$ in the resolved singularity).

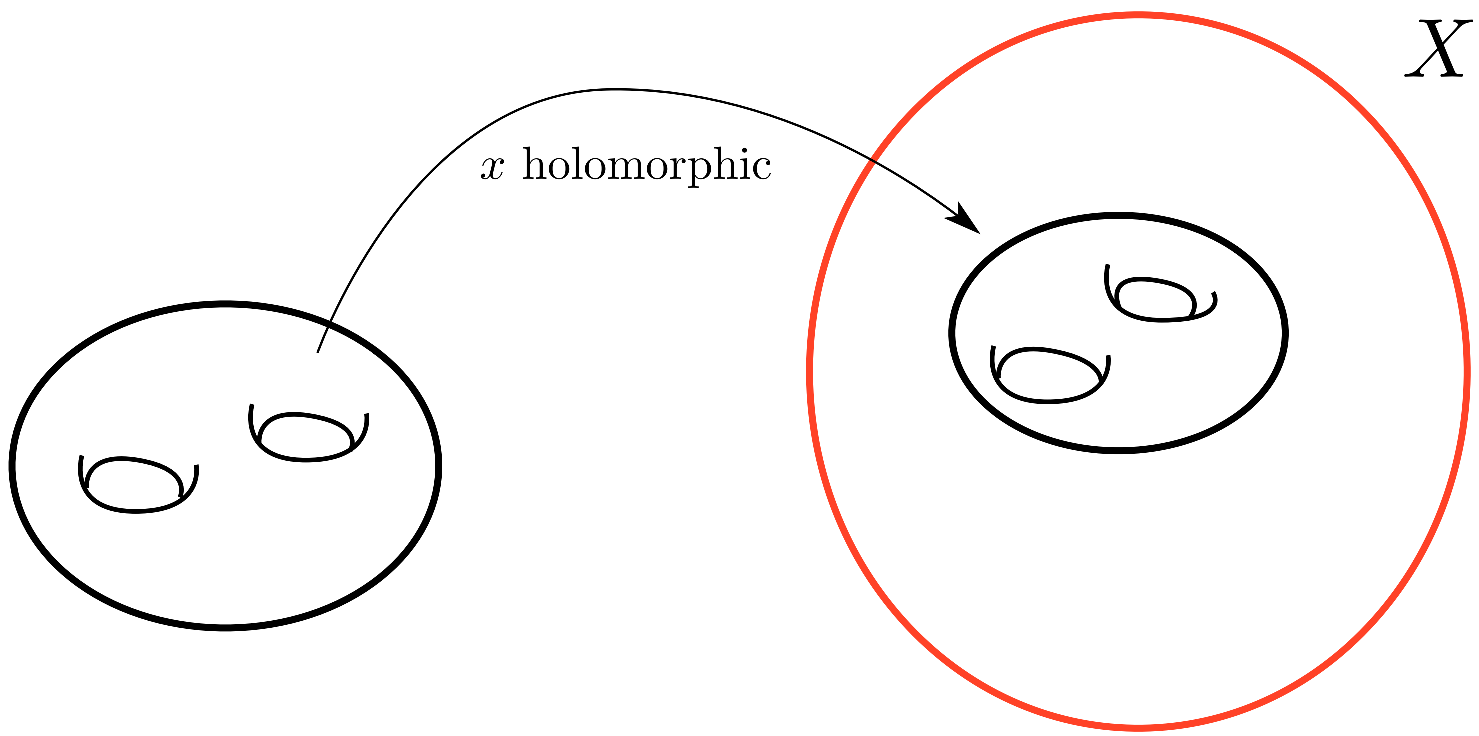






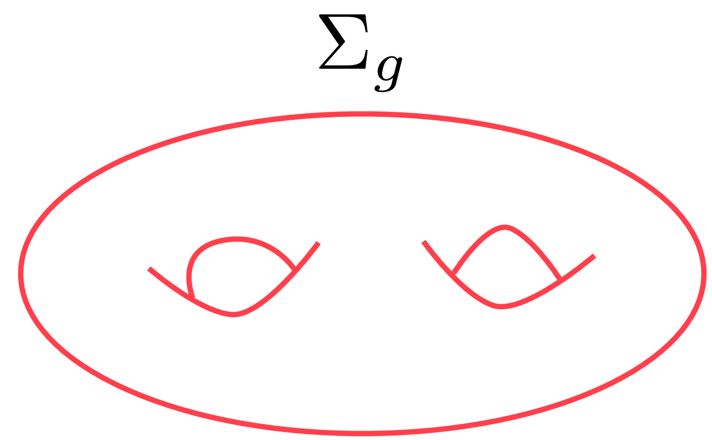
The A model

In the A model the free energy at genus g is exhausted by a semiclassical expansion around *instantons* of the 2d sigma model. These are *holomorphic maps* from a Riemann surface of genus g to the Calabi-Yau



These amplitudes encode an *enormous* amount of geometric information:

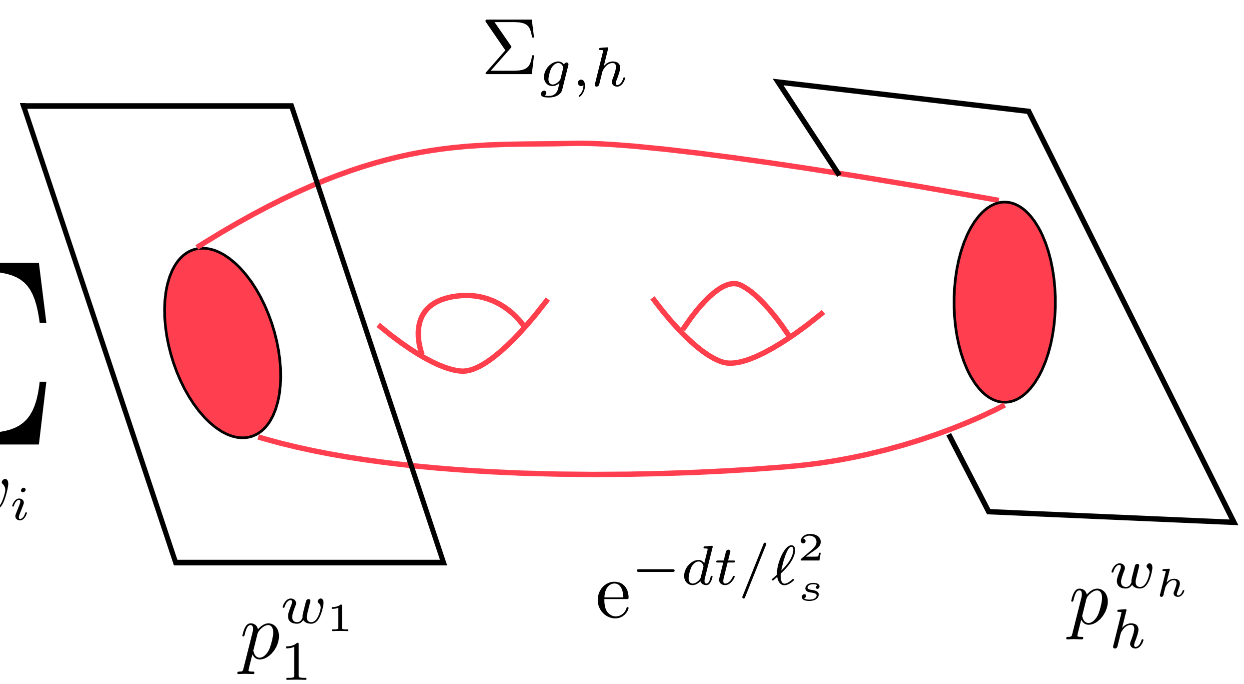
$$F_g(t) = \sum_d N_{d,g} \mathrm{e}^{-dt/\ell_s^2} = \sum_d$$



$$\mathrm{e}^{-dt/\ell_s^2}$$

It is also possible to add D-branes, in the form of *Lagrangian submanifolds*, and compute the corresponding open string amplitudes

$$F_{g,h}(t, p_1, \cdots, p_h) = \sum_{d,\, w_i}$$



The total free energy is given by

$$F(t, g_s) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$$

Notice that this theory has *two* quantum parameters:

- In the *worldsheet*, $\hbar_{\rm ws} = \ell_s^2$ and since it appears in the combination t/ℓ_s^2, weak coupling is *large size* in the target geometry
- In *spacetime*, $\hbar_{\rm st} = g_s$, the string coupling constant

The semiclassical expansion of the A model is a weak coupling expansion: it is only valid for large sizes and small string coupling constant.

Nonperturbative structure at small distances

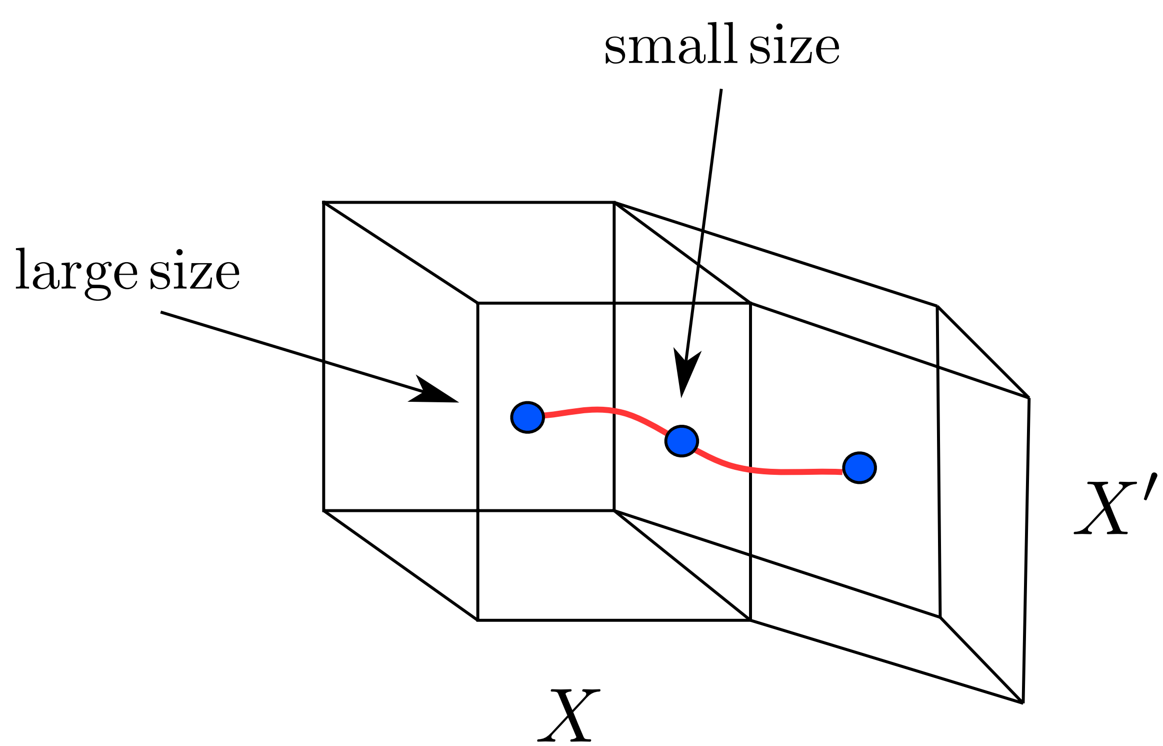
Thanks to mirror symmetry, much is known about the non perturbative structure of the total free energy as a function of ℓ_s

Main result: $F_g(t)$ is an *analytic function* of $Q = \mathrm{e}^{-t/\ell_s^2}$ at the origin (i.e. it has a finite radius of convergence $Q_* = \mathrm{e}^{-t_*/\ell_s^2}$).

The point t_* is called the *conifold* and at this point there is a *phase transition* characterized by the universal critical behavior [BCOV, Ghoshal-Vafa]

$$F_g(t) \sim \frac{B_{2g}}{2g(2g-2)} \mu^{2-2g}, \quad \mu \sim t - t_*$$

Consequence: there is a very rich phase structure in t-space [Witten, Aspinwall-Greene-Morrison...]. By studying the small size regime, we can discover for example *spacetime topology change* in string theory



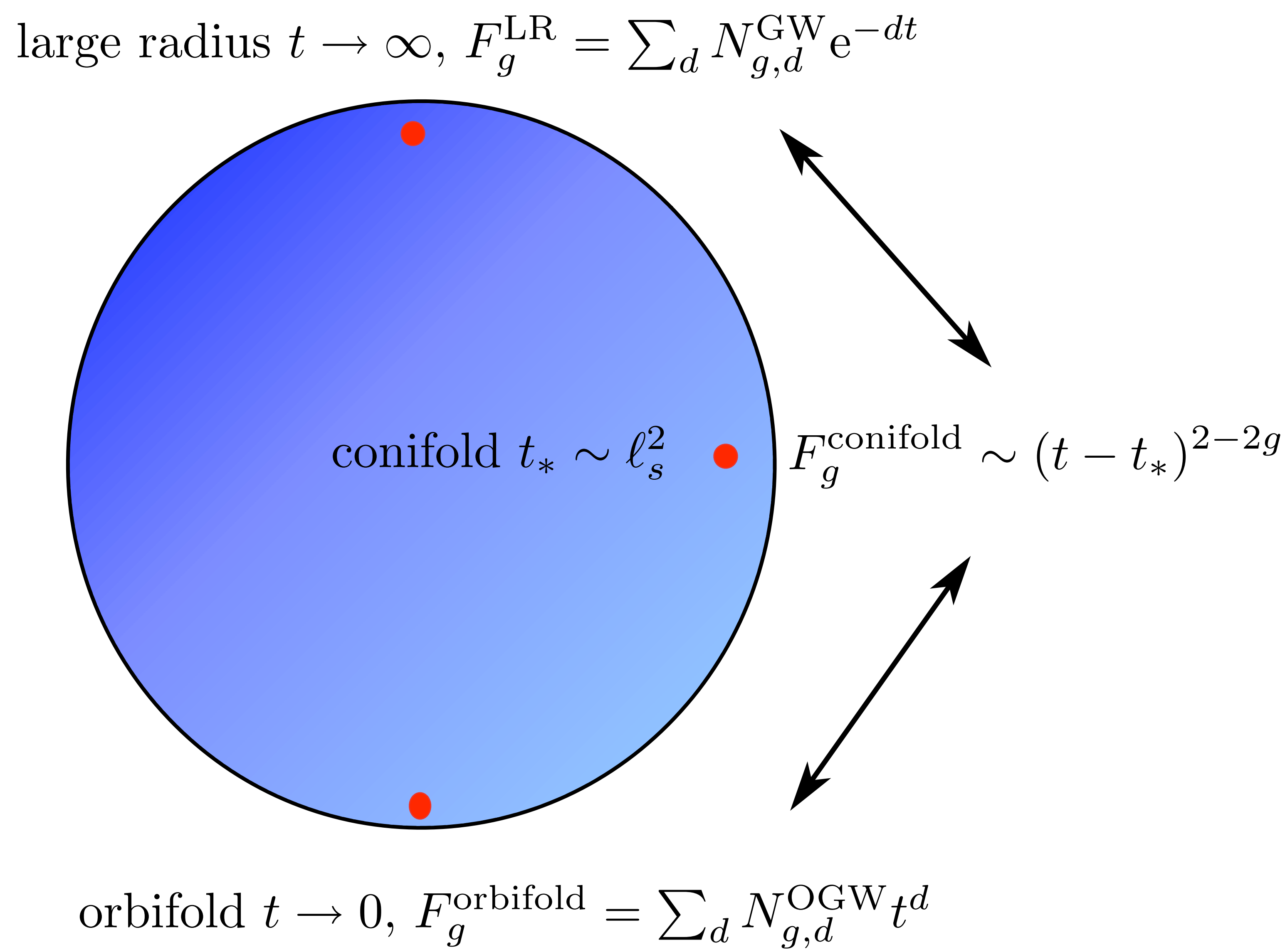
Global structure and modularity

There is a *modular group* acting on topological string amplitudes. In the toric case, this group is $\mathrm{Sp}(2g, \mathbb{Z})$, where g is the genus of the spectral curve

This acts on the standard way on the modular parameter of the curve:

$$\tau = \frac{\partial^2 F_0}{\partial t^2} \to \frac{\alpha \tau + \beta}{\gamma \tau + \delta}$$

The modular group relates different descriptions of the amplitudes. Typically, in different regions of the CY moduli space, some descriptions are more appropriate than others.

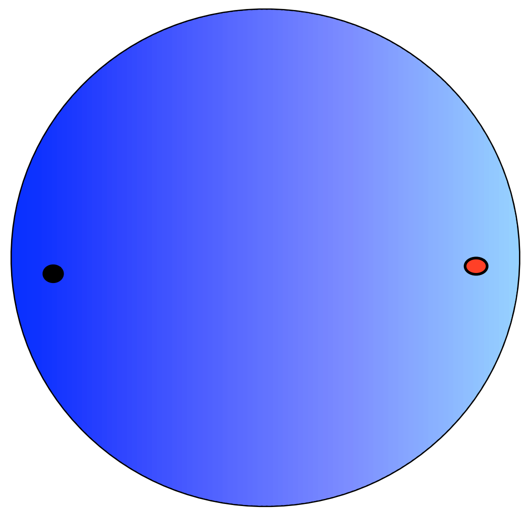


There are many genus g amplitudes, related by suitable actions of the symplectic group. At every special point (large radius, ...) there is a *preferred frame* and a set of parameters t where the amplitudes $F_g(t)$ have a good expansion (recall SW theory!). They define generalizations of GW theory

The relation between large and small distances can be regarded as an example of *strong/weak coupling interpolation*

In the case of the resolved conifold the interpolation is simply analytic continuation

$t = 0$



$t = \infty$

$$\frac{t^3}{12} - \frac{t^4}{288} + \cdots$$

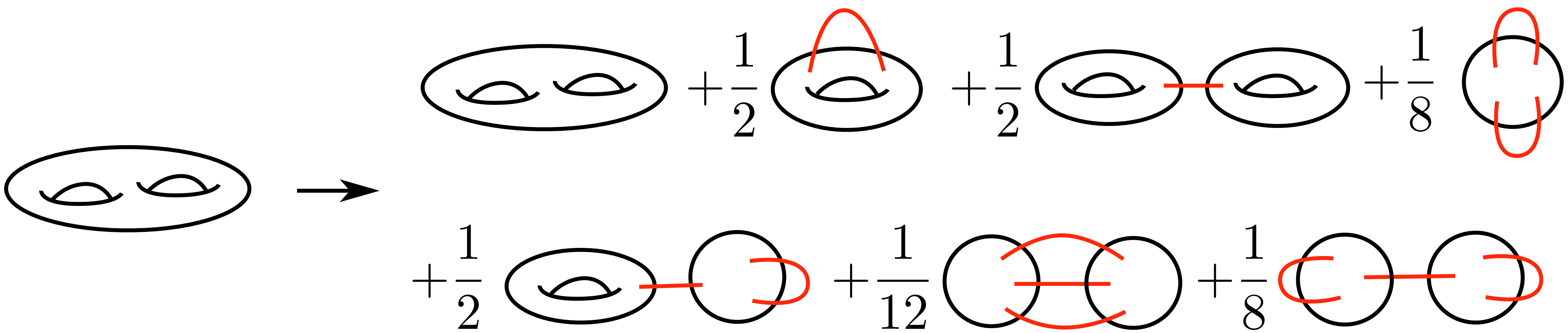
$$\mathrm{e}^{-t} + \frac{1}{8}\mathrm{e}^{-2t} + \cdots$$

$$\mathrm{Li}_3(\mathrm{e}^{-t})$$

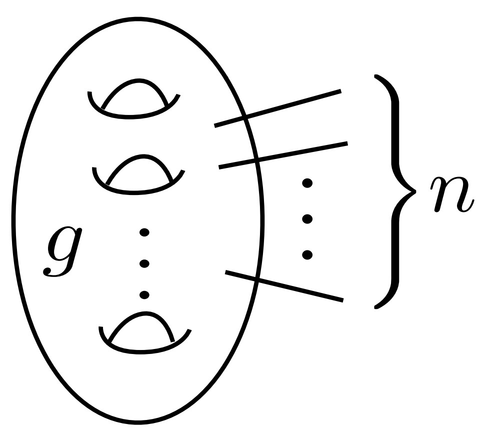
In general we have analytic continuation *plus* modular transformation

Modular properties

With respect to the modular group, the topological string amplitudes (which are *holomorphic*) transform with shifts, so they are only *quasi-modular* [BCOV,ABK].



Graphic notation:



$= F_g^{(n)}$

$= -\frac{1}{2\pi \mathrm{i}}(\gamma\tau+\delta)^{-1}\gamma$

The holomorphic anomaly

Almost modular forms can be promoted to full modular objects by introducing a *non-holomorphic dependence*

$$F_g(t) \to F_g(t, \bar{t})$$

The dependence on $\bar{t}$ can be deduced from worldsheet arguments [BCOV] and gives the famous *holomorphic anomaly equations*

$$\partial_{\bar{t}^I} F_g = \frac{1}{2} \overline{C}_{\bar{I}}^{JK} \Big(D_J D_K F_{g-1} + \sum_{h=1}^{g-1} D_J F_h D_K F_{g-h} \Big)$$

... so far the most general method to actually compute $F_g(t)$. They miss however holomorphic information (the *holomorphic ambiguity*). Supplemented with appropriate data at the "special points" in moduli space they can be sometimes solved to high genus [Huang-Klemm-Quackenbush]

Recap, goals and problems

For fixed genus, we understand quite well the topological string amplitudes, and in particular their nonperturbative properties as a function of the Kahler/complex moduli: they are analytic functions with modular properties (like elliptic functions).

Pragmatically, we would like to compute the topological string amplitudes at all genus. *Is the topological string exactly solvable on a generic CY?*

Like in any string theory, the genus expansion is only defined perturbatively. Are there *nonperturbative effects* that can be computed (in the string coupling constant)? Are there *nonperturbative definitions* of the theory?

We will address the last issue from a conservative point of view, i.e. by making use "as much as possible of the important pieces of information contained in the coupling constant expansion" (G. 't Hooft, 1979)

Let us then recall two basic facts in quantum theories:

- *perturbation theory leads, generically, to divergent series:*

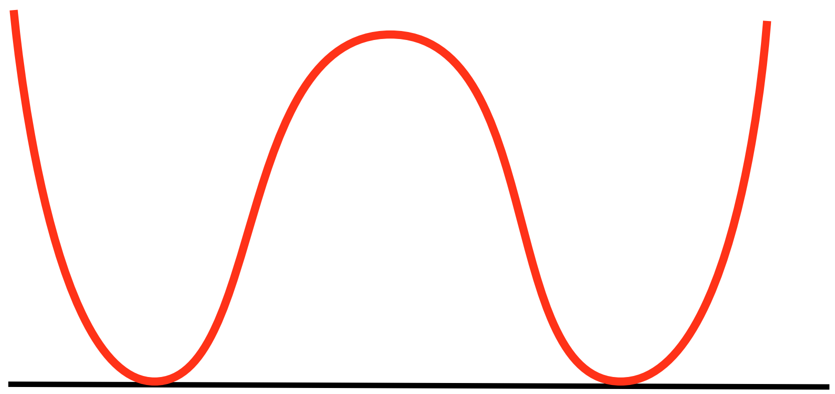
$$E = \sum_n a_n g^n, \qquad a_n \sim n! A^{-n}$$

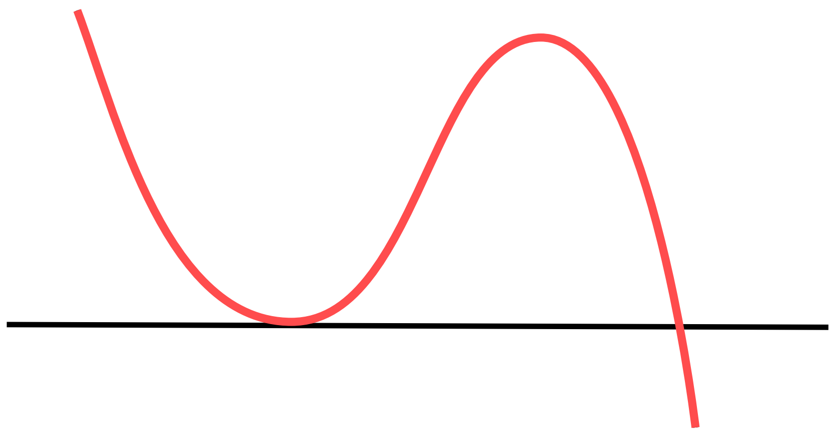
- *nonperturbative effects can be "discovered" by looking at perturbation theory at large order:*

$$E = E_{\rm p} + \mathcal{O}({\rm e}^{-A/g})$$

These effects are exponentially small corrections to the semiclassical expansion, so they are still weak coupling effects

Instantons in quantum mechanics





$$E_+ - E_- = \mathcal{O}(\mathrm{e}^{-A/g}) \qquad\qquad \mathrm{Im}\, E = \mathcal{O}(\mathrm{e}^{-A/g})$$

In both cases, the perturbative series is divergent, and resummation is *ambiguous* (in technical parlance, the series is not *Borel summable*). Nonperturbative effects change qualitatively the physics: In the first example, they *restore a symmetry* which is broken in perturbation theory and lift the degeneracy. In the second example, they trigger *false vacuum decay*

In the first case, there is a clear nonperturbative definition of the theory (nonambiguous ground state energy). In the second case, there is no clear nonperturbative definition

These nonperturbative effects are due to *instantons*. By including them, we are taking into account *different semiclassical backgrounds* which were not included in our original description.

What happens in (topological) string theory?

One expects similar (but stronger) nonperturbative effects [Shenker]

$$F_g(t) \sim (2g)!(A(t))^{-2g} \Rightarrow \mathcal{O}(\mathrm{e}^{-A(t)/g_s})$$

found experimentally in non-critical string theory, identified as D-instanton effects later on [Polchinski 1994]

Large N duals

If the string theory we are considering has a large N dual, we might be able to compute these effects reliably. We will focus on topological strings whose large N dual is a matrix model/ Chern-Simons gauge theory

We recall that matrix models are defined by integrals over Hermitian matrices

$$F = \log \int dM \, e^{-\frac{1}{g_s} \mathrm{tr} \, V(M)}$$

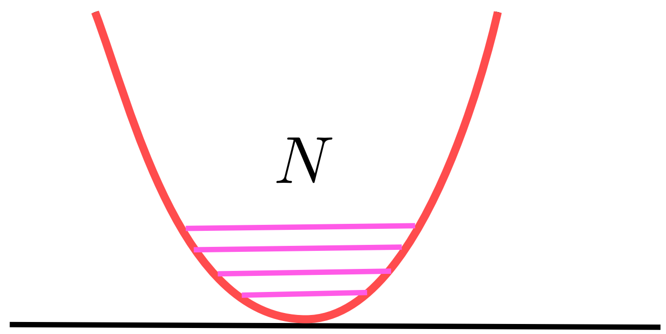
This free energy has an asymptotic expansion of the form

$$F = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}, \qquad t = N g_s$$

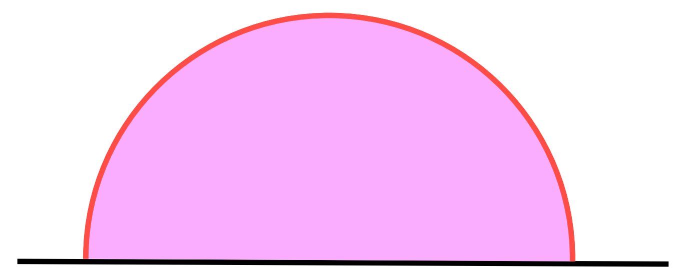
't Hooft parameter

Master fields and algebraic curves

At large N, the matrix eigenvalues reach an equilibrium distribution and they sit along intervals in the complex plane. Their density is supported on these intervals.



$$V(x) = \frac{x^2}{2}$$

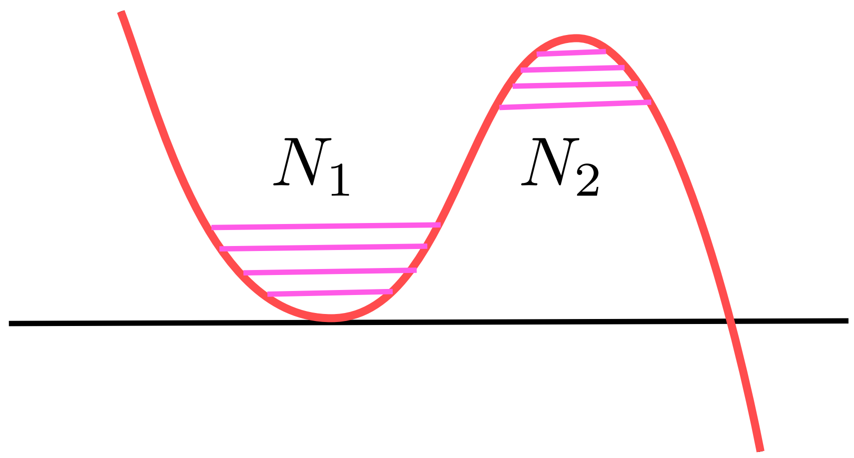


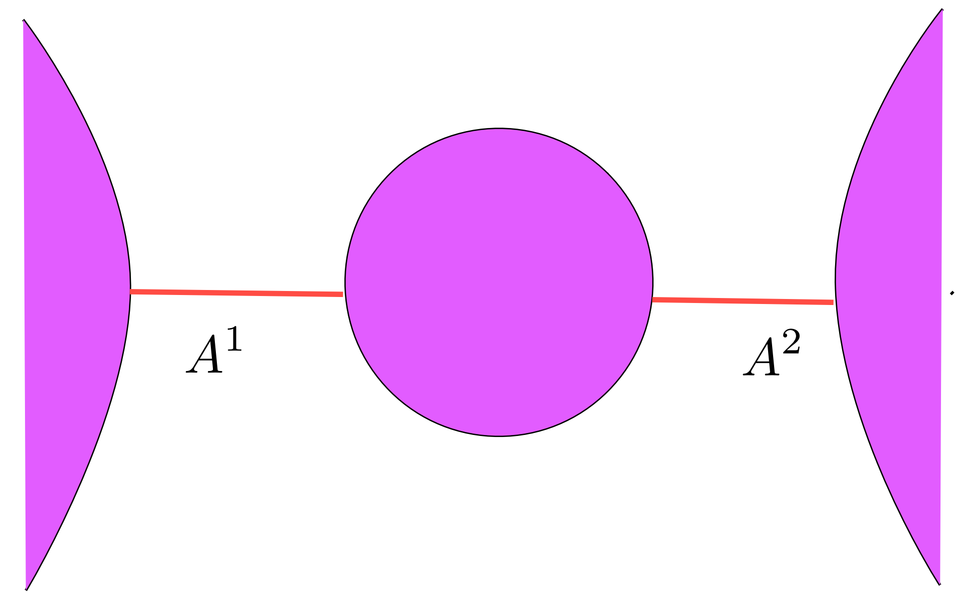
$$\rho(\lambda) = \frac{1}{2\pi t}\sqrt{4t - \lambda^2}$$

If we have s extrema of the potential, the general saddle of the matrix model is described by s sets of N_i eigenvalues

$$t_i = g_s N_i \longleftarrow$$

partial 't Hooft parameters





Equivalently, the intervals can be regarded as branch cuts of a *spectral curve* $y(x)$ characterizing the model. The density of eigenvalues is given by

$$\rho(x) = \frac{1}{2\pi t} \mathrm{Im}\, y(x)$$

Given the spectral curve only, the large N expansion of the free energy and correlators can be computed explicitly and recursively at all orders [Ambjorn et al, Eynard et al]

Chern-Simons theory

Another U(N) theory we will use is Chern-Simons theory. This is an exactly solvable topological gauge theory in 3d with action

$$S = \frac{1}{g_s}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

On some special 3-manifolds, this theory is essentially equivalent to a matrix model [M.M.] and we will use the matrix model language/description