NON-PERTURBATIVE ASPECTS OF THE TOPOLOGICAL STRING

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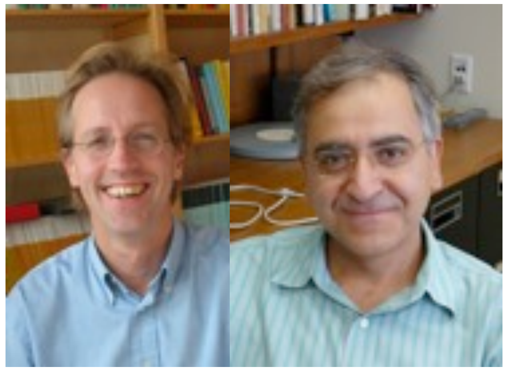
[M.M.-Schiappa-Weiss, 0711.1954 & 0809.2619]
[M.M., 0805.3033]
[B. Eynard-M.M., 0810.4273]
related: [S. Garoufalidis-M.M., 0811.1195]

A CONSERVATIVE POINT OF VIEW II

Large N duals



Basic example: Dijkgraaf-Vafa CY backgrounds

$$Z = \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} V(M)}$$

master field

type B TS on

$$u^2 + v^2 + y^2 - (V'(x))^2 + f(x) = 0$$

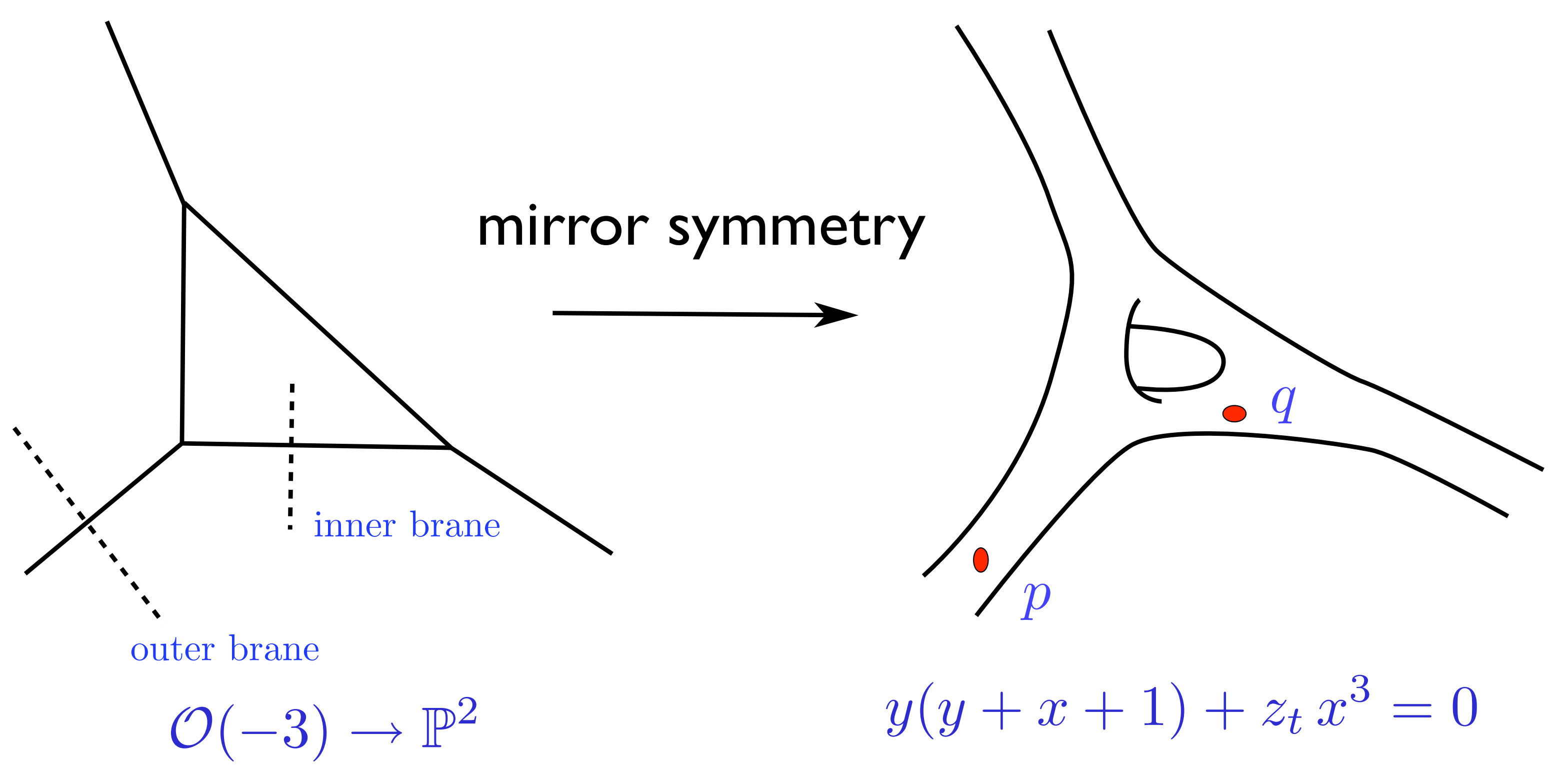
$$y^2 = (V'(x))^2 - f(x)$$ spectral curve

$$\rho(\lambda) = \frac{1}{2\pi t} \mathrm{Im}\, y(\lambda)$$

1/N expansion $\log Z = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$ $\longleftrightarrow$ perturbative topological string amplitudes

$$F_g^{\mathrm{MM}}(t_i = g_s N_i) = F_g^{\mathrm{TS}}(t_i = \mathrm{moduli})$$

Toric backgrounds



A-model: perturbative topological string amplitudes given by GW invariants

(Remodeling the) B-model: residue calculus on the mirror/spectral curve [M.M., BKMP]

In principle, no need of explicit matrix model. For some CYs (resolved conifold, local P1xP1): Chern-Simons gauge theory description [Gopakumar-Vafa, AKMV] and CS matrix model

Remarks on large N duals

- *Analyticity* of TS amplitudes is here equivalent to analyticity in the ‘t Hooft parameters, expected in large N theories without renormalons

- *Nonperturbative effects* in the string coupling correspond to effects of the type e^{-N} in the large N dual

- *Quantum theory on the spectral curve*: unified description of matrix models, noncritical strings, toric backgrounds and CS theory

Guidelines for nonperturbative computations

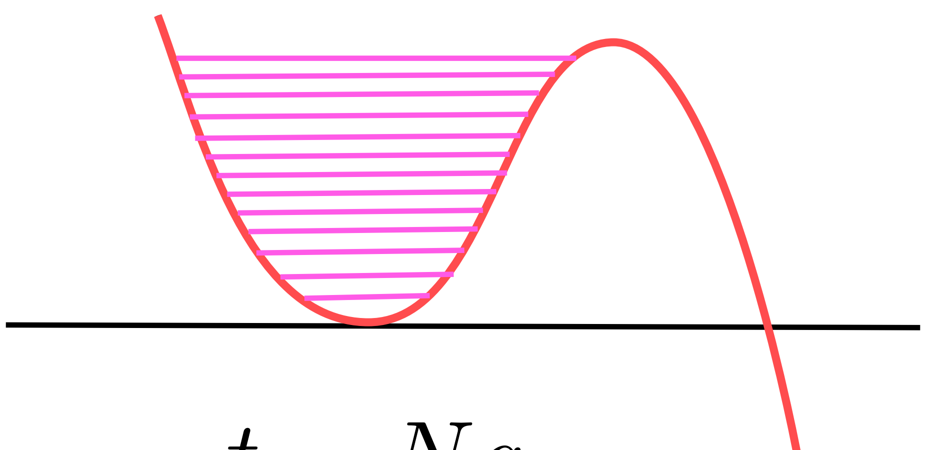
- *Explicit checks with the large order behavior of string/large N perturbation theory*: the instanton amplitude should control the large order behavior of the perturbative series

- *Symmetries*: nonperturbative effects might restore symmetries which are broken in perturbation theory, like in the double-well potential.

A toy model: instantons in 2d gravity

2d gravity means here the (2,3) minimal CFT coupled to gravity

- *Continuum description* in terms of Liouville and matter
- *Large N description* in terms of a doubly-scaled matrix model
- *TFT description* in terms of intersection theory on Deligne-Mumford moduli space
- *Integrable model description* in terms of KdV/Painleve I



$t = N g_s$

$$F_g(t) \sim c_g (t - t_c)^{5/2(g-1)}$$

$$F_{\rm ds}(z) = \sum_g c_g z^{-5g/2}$$

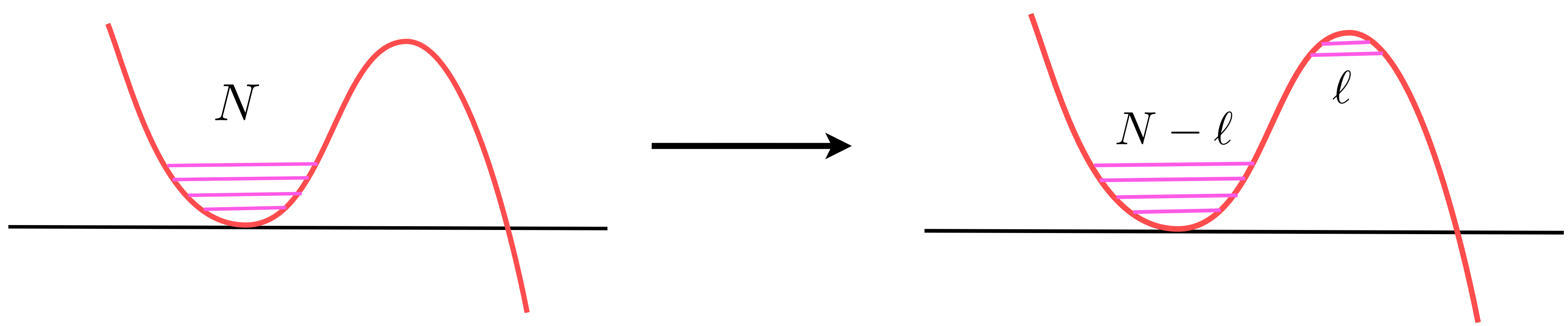
$$u = F''_{\rm ds}(z) \text{ satisfies PI: } -\frac{1}{6}u'' + u^2 = z$$

• In the *integrable model description*, instantons are obtained by solving PI with a *trans-series* ansatz:

$$u = u^{(0)}(z) + \sum_{\ell \geq 1} C^{\ell} \mathrm{e}^{-\ell A z^{5/4}} u^{(\ell)}(z), \qquad z^{5/4} = \frac{1}{g_s}$$

The trans-series involves *two small parameters*: g_s and e^{-1/g_s} (number of loops/number of instantons)

• In the *matrix model description*, instantons correspond to (the double-scaling limit of) *eigenvalue tunneling* [David 91,93]



• In the *continuum description*, instantons correspond to *D-instantons/ZZ branes* [Martinec, Alexandrov-Kazakov-Kutasov]. However, *no known description* of the instantons in terms of TFT/intersection theory (TFT analogue of ZZ brane?)

Check with large order: $c_g \sim (2g)! A^{-2g}, \quad A = \frac{8\sqrt{3}}{5}$

But we can be much more precise here:

$$c_g = S\,\Gamma\left(2g - \frac{5}{2}\right) A^{-2g} \left\{1 + \frac{a_1}{g} + \frac{a_2}{g^2} + \cdots \right\}$$

Stokes constant (one instanton at one-loop)

one instanton at two-loops

one instanton at three-loops

We have a hierarchical structure in which the asymptotics of the ℓ-instanton series is related to the $(\ell \pm 1)$-instanton: *resurgence* [Ecalle, 1980]. Interestingly, resurgence predicts the existence of "exotic" sectors, crucial in the understanding of the asymptotics, but with *no known matrix model interpretation*

[Garoufalidis-Its-Kapaev-M.M., to appear]

The nonperturbative ambiguity

We have computed a genuine nonperturbative effect, but we are still not producing the semiclassical expansion of the *exact* answer, since C (the strength of the instanton corrections) cannot be determined: this is the *non-perturbative ambiguity.*

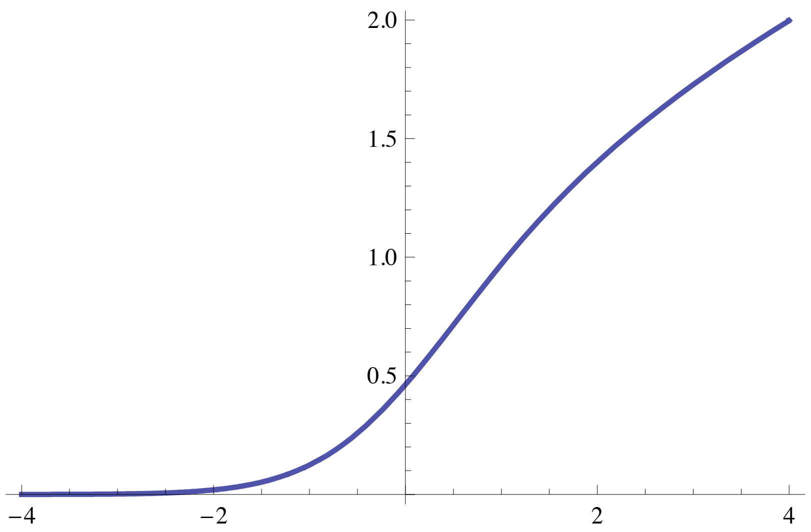
In general, the ambiguity is due to the need of choosing an integration contour in the matrix/path integral (this has been argued to be relevant in QFT as well [Guralnik]). In some cases one can fix this ambiguity by using nonperturbative input.

Fixing the nonperturbative ambiguity I

The specific heat of the noncritical minimal *superstring* is described (at zero RR flux) by the Hastings-McLeod solution to PII [MKS]

$$u'' - 2u^3 + 2zu = 0$$

$$u_{\rm HM} = u^{(0)} - \frac{S^2}{8} u^{(2)} + \cdots$$



The strength of instanton corrections can be fixed [M.M.] and we then obtain the *exact semiclassical expansion* of the nonperturbative solution

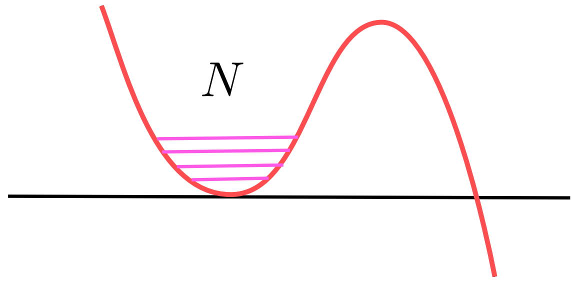
The details involve Borel resummation and a nonpertubative cancellation mechanism which has been discussed in renormalon physics and in the double-well potential

Instantons in the full matrix model

We will compute instanton effects in topological string theory by computing them in large N matrix model duals, so we need to understand eigenvalue tunneling in matrix models before taking the double-scaling limit. We first consider *one-cut* matrix models

perturbative partition function

$$Z^{(0)}(t, g_s) = \int_{\text{one cut}} \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} \mathrm{Tr}\, V(M)}$$

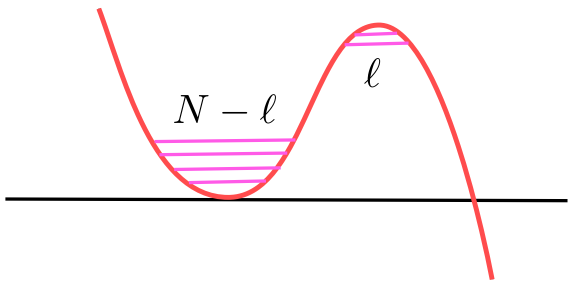


1/N expansion

$$F^{(0)}(t, g_s) = \log Z^{(0)}(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

The total partition function will include instanton corrections to the perturbative partition function, coming from eigenvalue tunneling:

$$Z(t, g_s) = Z^{(0)}(t, g_s) + \sum_{\ell \geq 1} \mathrm{e}^{-\ell A(t)/g_s} C^\ell Z^{(\ell)}(t, g_s)$$



Traditionally there are two ways to compute the perturbative partition function in the 1/N expansion: by using *orthogonal polynomials* or by using *saddle-point methods*. One can use both approaches to compute instanton corrections

We expect again a connection between the large order behavior of the 1/N expansion and these instanton effects

$$F_g(t) \sim (2g)!(A(t))^{-2g}$$

Instantons and difference equations

The method of orthogonal polynomials is closely related to an *integrable model description*. The one-cut model is obtained by solving a difference equation of the Toda type. For example, for the quartic matrix model one has the off-critical analogue of PI:

$$R(t,g_s)\Big\{1-\frac{\lambda}{12}\big(R(t,g_s)+R(t+g_s,g_s)+R(t-g_s,g_s)\big)\Big\}=t$$

$$F(t+g_s)+F(t-g_s)-2F(t)=\log R(t,g_s),$$

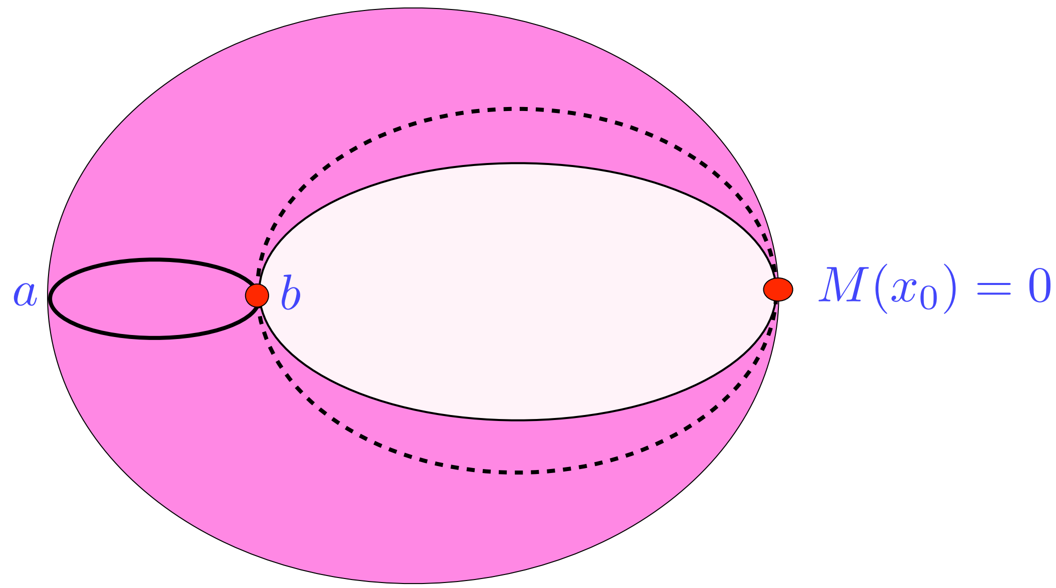
The instanton amplitudes are obtained, as in the critical case, by using a trans-series ansatz for R and F [M.M.]. We should expect resurgence properties

$$R(t,g_s)=\sum_{g\geq 0} R_k^{(0)}(t)g_s^{2k}+\mathcal{O}(\mathrm{e}^{-A(t)/g_s})$$

Instantons and the spectral curve

Alternatively, one can solve the model by spectral curve/saddle point methods. In the one-cut case, this curve is singular

$$y(x) = M(x)\sqrt{(x-a)(x-b)}$$



Instanton amplitudes can also be computed by *using spectral curve data only*. For example,

$$A(t) = \int_b^{x_0} y(x)\mathrm{d}x$$

One can obtain explicit formulae for the loop corrections around the instanton in terms of spectral curve data [M.M.-Schiappa-Weiss '07]

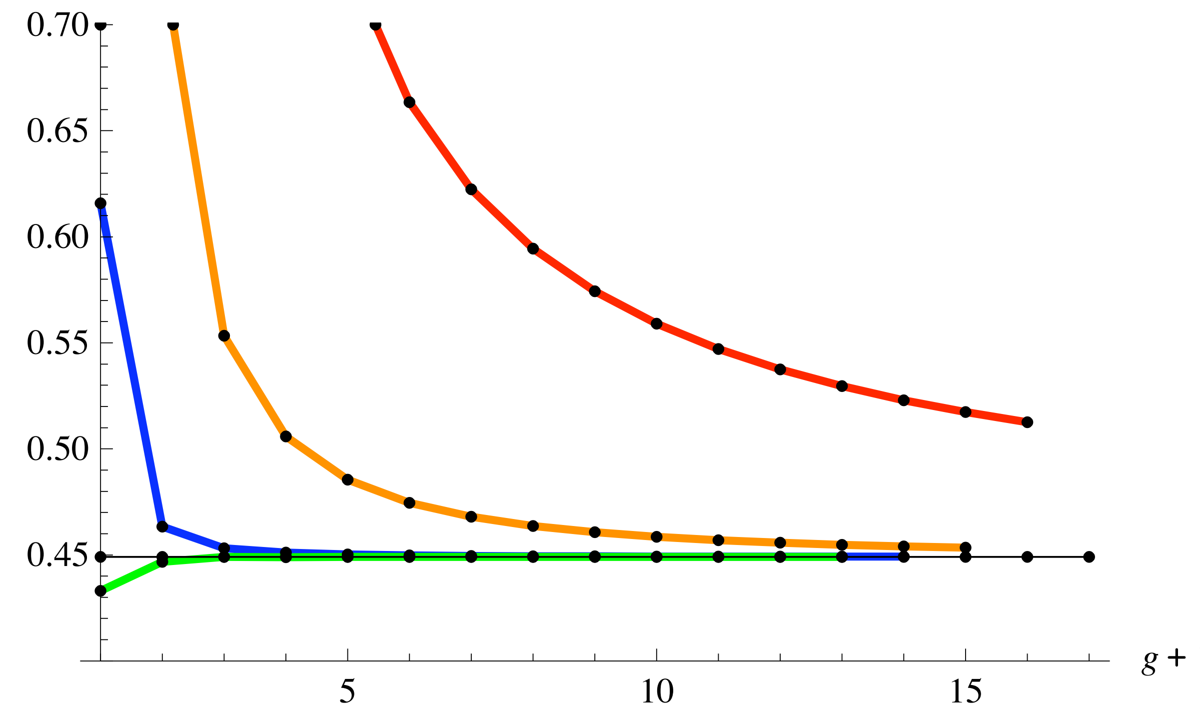
Examples of one-cut models

Of course, we can apply these results to matrix models, but we are interested in topological strings described either by difference equations or one-cut spectral curves

family of Calabi-Yau's $\mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1, \quad p \geq 2$

$p \to \infty$ chiral 2d YM/Hurwitz/Toda

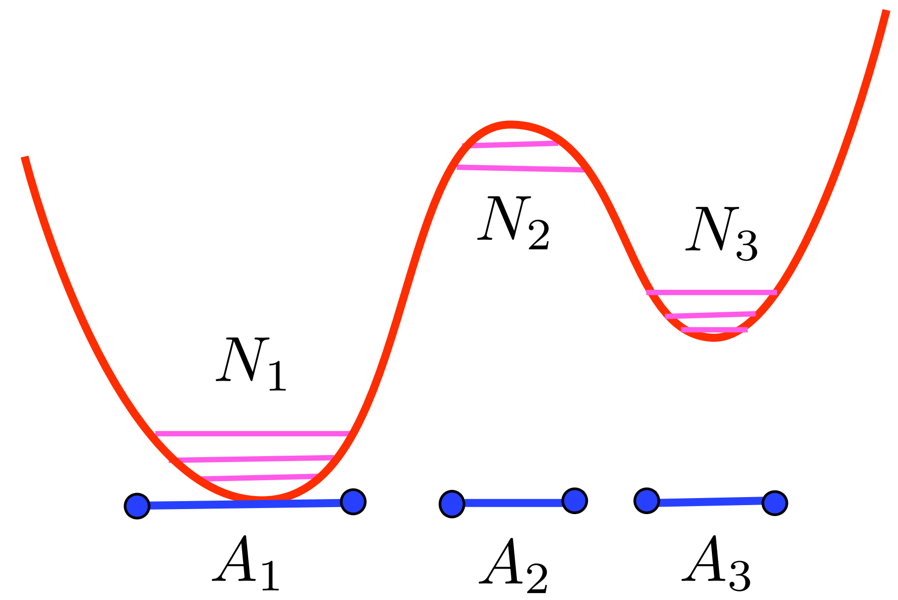
The nonperturbative effects computed in this way are *testable* through the connection with large order behavior



$2g(F_g(t)/F_{g+1}(t))^{\frac{1}{2}}$ and its Richardson transforms for a fixed value of t, as compared to the instanton action (horizontal line)

Multi-cut matrix models

In order to understand more general topological string theories, we must consider multi-cut matrix models, in which *all extrema* are filled up (a lesson from Dijkgraaf and Vafa).



In large N dualities, the perturbative TS partition function corresponds to a *fixed*, *generic filling fraction* of the matrix model. We call this a *background*.

fix a background

$$Z_{\mathrm{MM}}(N_1, \cdots, N_p) = Z_{\mathrm{TS}}(t_1, \cdots, t_p)$$

TS partition function

$$t_i = g_s N_i$$

Given a *reference background*, any other background can be regarded as an instanton:

$$\frac{Z(N_1', \ldots, N_d')}{Z(N_1, \ldots, N_d)} \sim \exp\left\{ -\frac{1}{g_s} \sum_{I=1}^{s} (N_I - N_I') \frac{\partial F_0(t_I)}{\partial t_I} \right\}.$$

We don't see other backgrounds in string perturbation theory!

In particular, we can think about the instantons of the one-cut matrix model as limiting cases of a *two-cut* matrix model. This gives a powerful method to compute their amplitudes and in particular gives regularized formulae for ZZ brane amplitudes in 2d gravity [M.M.-Schiappa-Weiss '08]

We can incorporate all backgrounds by *summing over all filling fractions*

$$Z(C_1, \ldots, C_s) = \sum_{N_1+\cdots+N_s=N} C_1^{N_1} \cdots C_s^{N_s} Z(N_1, \ldots, N_s).$$

We call this the *nonperturbative partition function* of the matrix model. If we choose a reference background, we obtain the perturbative partition function plus instanton corrections/ eigenvalue tunnelings, similar to the total Z we studied in the one-cut problem. Here however we consider a generic background and both the stable and unstable directions [Bonnet-David-Eynard, Eynard].

Agrees withvarious previous proposals , although the interpretation is slightly different: inclusion of fermion fluxes in [ADKMV], I-brane partition function of [Dijkgraaf-Hollands-Sulkowski-Vafa].

Explicit form of Z at large N

Given any spectral curve, the nonperturbative partition function is given by an all-orders g_s expansion (cf. [Dijkgraaf-Verlinde-Vonk])

$$Z_\Sigma(\mu,\nu;t,g_s) = Z_{\rm pert}(t,g_s)\Big\{\Theta_{\mu,\nu} + g_s\Big(\Theta'_{\mu,\nu}F'_1 + \frac{1}{6}\Theta'''_{\mu,\nu}F'''_0\Big)$$

$$+ g_s^2\Big(\frac{1}{2}\Theta''_{\mu,\nu}F''_1 + \frac{1}{2}\Theta''_{\mu,\nu}F'^2_1 + \frac{1}{24}\Theta^{(4)}_{\mu,\nu}F''''_0 + \frac{1}{6}\Theta^{(4)}_{\mu,\nu}F'''_0F'_1 + \frac{1}{72}\Theta^{(6)}_{\nu,\mu}F'''^2_0\Big) + \dots\Big\}$$

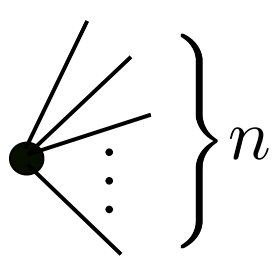
where

$$\Theta_{\mu,\nu}(u,\tau) = \sum_{n\in\mathbb{Z}^g} \mathrm{e}^{(n+\mu-t/g_s)u}\ \mathrm{e}^{\pi\mathrm{i}(n+\mu-t/g_s)\tau(n+\mu-t/g_s)}\ \mathrm{e}^{2\mathrm{i}\pi n\nu} \qquad u = \frac{1}{g_s}\frac{\partial F_0}{\partial t}$$

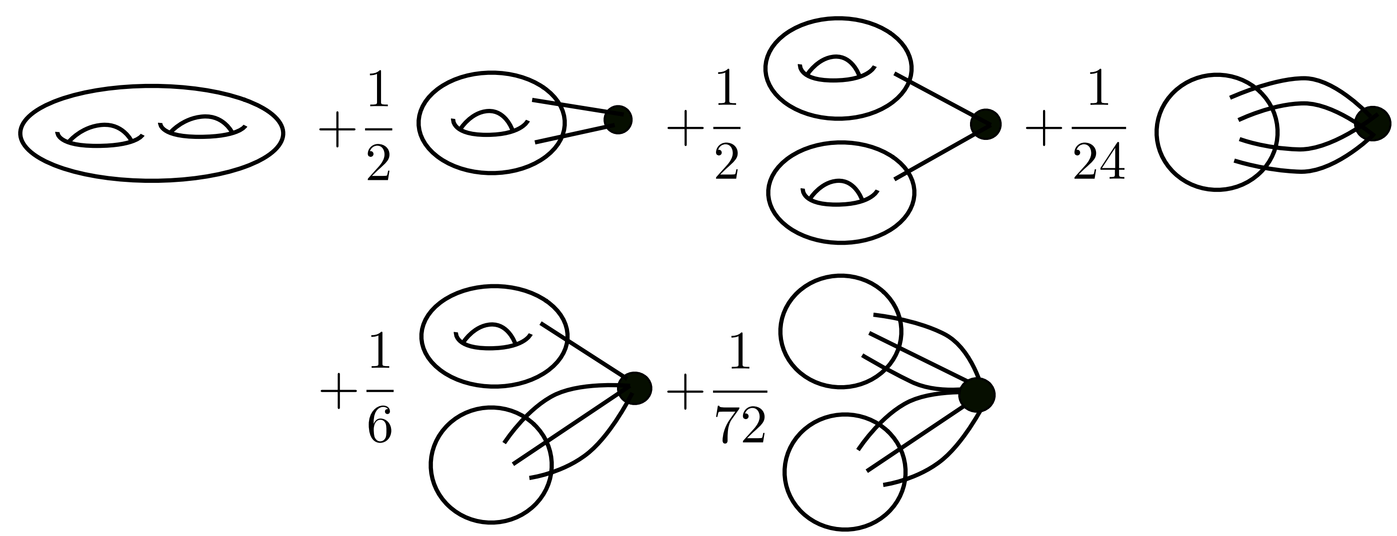
dual characteristic, no MM interpretation

nonperturbative ambiguities=characteristic

graphic representation



$$= \frac{\Theta^{(n)}}{\Theta}$$



This holomorphic anomaly has been interpreted in terms of background independence [Witten]. Since in the nonperturbative Z we have summed over all backgrounds, we should expect it to be modular and holomorphic.

In other words, if we add to the perturbative partition function (evaluated on a reference background) the sum over spacetime instantons *we should restore modularity*

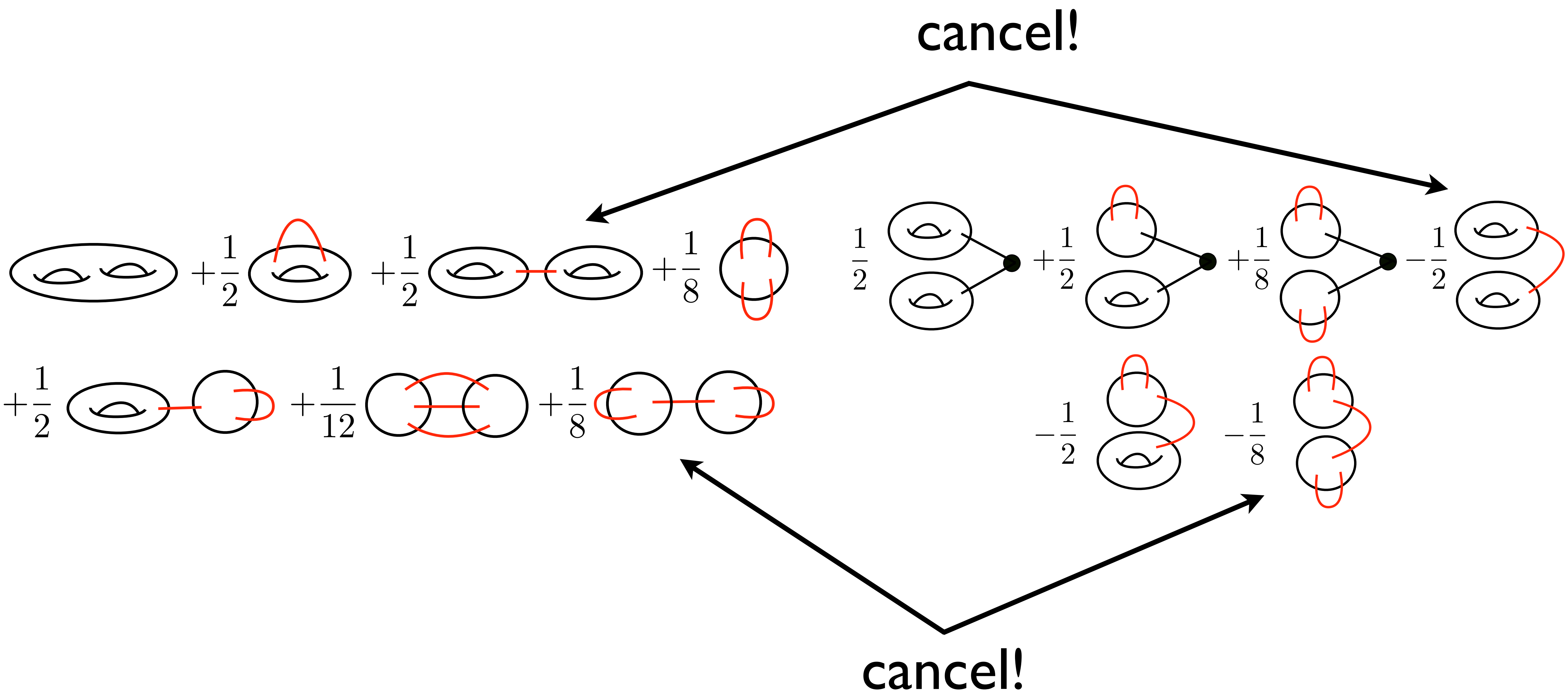
This looks impossible at first sight, since the breaking of modularity is a perturbative phenomenon, while instantons are nonperturbative. In fact, we must resum all terms in e^{-1/g_s} at each order in g_s. This resummation leads to theta functions, so there is a chance for modularity!

Modularity of Z

The nonperturbative partition function transforms as the partition function of a twisted fermion on the spectral curve:

$$\tilde{Z}_\Sigma(\tilde{\mu}, \tilde{\nu}; \tilde{t}, g_s) = \zeta\left[{\mu \atop \nu}\right](\Gamma) Z_\Sigma(\mu, \nu; t, g_s)$$

phase



Fixing the nonperturbative ambiguity II

In this formulation, the nonperturbative ambiguity becomes a characteristic in the theta function, which is not fixed *a priori*

However, if we have a *Chern-Simons dual* we can fix the value of the ambiguity/characteristic by comparing to the exact Chern-Simons partition function.

$$\text{\textit{sum} all backgrounds}$$

$$Z_{\rm CS}(\mathbf{S}^3/\mathbf{Z}_p; N, g_s) = \sum_{N_1+\cdots+N_p=N} C_1^{N_1} \cdots C_p^{N_p} Z_{\rm MM}(N_1, \cdots, N_p)$$

Notice that in this case the different TS backgrounds correspond to different topological sectors of the Chern-Simons theory (i.e. flat connections)

We might be able to construct *modular invariants* as in CFT

$$\sum_{(\mu,\nu)} |Z_\Sigma(\mu,\nu)|^2$$

OSV?

Conclusions

We have developed a formalism to understand instanton corrections in topological string theory, by using matrix model large N duals. We have tested these corrections with perturbative tools, following 't Hooft advice (large order behavior, restoration of modularity).

• Connection to large order behavior/resurgence in the generic multi-cut case [work in progress with Klemm and Rauch].

• Intriguing role of exotic sectors: they must be included for consistency. Related to anti-D-branes?

• D-brane/geometric interpretation of the instanton sectors.

• Intrinsic formulation of the nonperturbative partition function (cf. I-brane theory). Making it more concrete!