The AdS/CFT Correspondence and Sasaki-Einstein Geometry I: Overview

Dario Martelli (Swansea)

Based on work with:
Gauntlett, Maldacena, Sparks, Tachikawa, Waldram, Yau

Instituto Superior Técnico, Lisbon, Portugal,
June 29 - July 1, 2009

Outline

1. AdS/CFT correspondence and branes at singularities
2. Sasaki-Einstein geometry
3. Basic checks: symmetries, volumes
4. More advanced checks: moduli spaces and "counting" BPS operators
5. Volume minimisation and a-maximisation
6. Some examples
7. AdS_4/CFT_3 correspondence
8. Beyond the realm of Sasaki-Einstein geometry

The AdS/CFT correspondence

Maldacena conjecture

- $\text{AdS}_5 \times \mathbf{S^5}$ dual to $\mathcal{N} = \mathbf{4}$ $\mathbf{U(N)}$ super-Yang-Mills (1997)

The AdS/CFT correspondence

Maldacena conjecture

- $AdS_5 \times \mathbf{S^5}$ dual to $\mathcal{N} = \mathbf{4}$ $\mathbf{U(N)}$ super-Yang-Mills (1997)
- $AdS_4 \times \mathbf{S^7}$ dual to $\mathcal{N} = \mathbf{8}$ $\mathbf{U(N)_1} \times \mathbf{U(N)_{-1}}$ Chern-Simons-matter (ABJM 2008)

The AdS/CFT correspondence

Maldacena conjecture

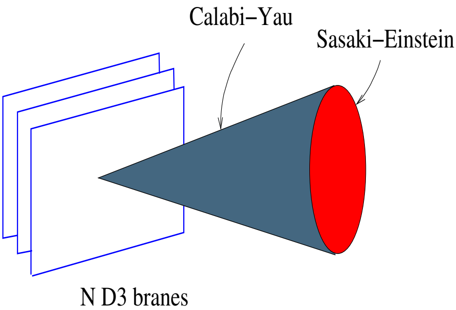
- $\mathrm{AdS}_5 \times \mathbf{S}^5$ dual to $\mathcal{N} = 4$ **U(N)** super-Yang-Mills (1997)
- $\mathrm{AdS}_4 \times \mathbf{S}^7$ dual to $\mathcal{N} = 8$ $\mathbf{U(N)_1} \times \mathbf{U(N)_{-1}}$ Chern-Simons-matter (ABJM 2008)

Our aim:

- Study the AdS/CFT correspondence for $\mathbf{0} < \mathcal{N} < \mathcal{N}_{\mathbf{maximal}} \to$ beautiful interplay with geometry

D3-branes at cone singularities

- Supersymmetric gauge theories can be engineered placing **N D3** branes transverse to a three-fold conical singularity $\mathbf{X_6}$



- For AdS/CFT applications we require that there is a Ricci-flat cone metric $\mathbf{ds^2(X_6) = dr^2 + r^2 ds^2(Y_5)}$ [Sometimes it does not exist [Gauntlett,DM,Sparks,Yau]]

D3-branes at cone singularities

Gravity solutions

- The "near-horizon" type IIB supergravity solution is: $\mathrm{AdS}_5 \times \mathbf{Y_5}$
- If $\mathbf{Y_5} = \mathbf{S^5}/\Gamma$ is an orbifold, various fractions of supersymmetry can be preserved
- If $\mathbf{Y_5}$ is a smooth Sasaki-Einstein manifold the solution is non singular and preserves $\mathcal{N} = \mathbf{1}$ supersymmetry (8 supercharges)

D3-branes at cone singularities

Gravity solutions

- The "near-horizon" type IIB supergravity solution is: $AdS_5 \times \mathbf{Y_5}$
- If $\mathbf{Y_5} = \mathbf{S}^5/\Gamma$ is an orbifold, various fractions of supersymmetry can be preserved
- If $\mathbf{Y_5}$ is a smooth Sasaki-Einstein manifold the solution is non singular and preserves $\mathcal{N} = 1$ supersymmetry (8 supercharges)

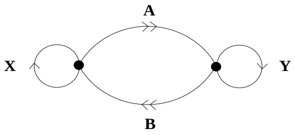
Gauge theories

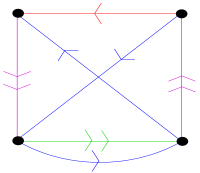
- When $\mathbf{X_6} = \mathbb{C}^3/\Gamma$ the gauge theory is the orbifold projection "$\mathcal{N} = 4/\Gamma$": a "quiver" gauge theory with gauge group $\mathbf{U(N_1)} \times \cdots \times \mathbf{U(N_n)}$ [Douglas-Moore]
- When $\mathbf{X_6} = \mathbf{C(Y_5)}$ it is harder to identify the gauge theory. If the singularity is toric there are powerful techniques (e.g. brane tilings) for deriving the gauge theory. These are again of quiver type

Supersymmetric gauge theories

Quivers

- Constructed from microscopic open string d.o.f. on D3-branes
- $\mathcal{N} = 1$ SYM with gauge group $\mathbf{G} = \mathbf{U(N_1)} \times \cdots \times \mathbf{U(N_n)}$
- Coupled to bi-fundamental chiral fields $\mathbf{X_i}$ ("matter")
- Full Lagrangian $\mathcal{L} = \mathcal{L}_{\mathbf{YM}} + \mathcal{L}^{\mathbf{matter}}_{\mathbf{kin}} + \mathbf{W}$

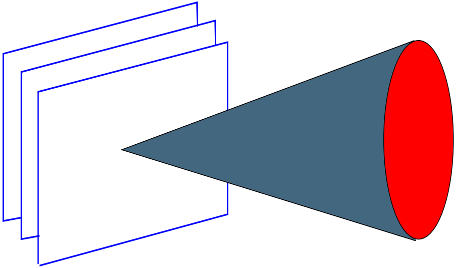




node $= \mathbf{U(N)}$ arrow $= (\mathbf{\bar{N}}, \mathbf{N})$ chiral field $\mathbf{X_i}$ $\mathbf{W}$= polynomial in $\mathbf{X_i}$

M2-branes at cone singularities

- Supersymmetric (gauge?) theories should be obtained placing **N M2** branes transverse to a four-fold conical singularity $\mathbf{X_8}$ [reduce to D2 in the type IIA limit]



N M2 branes

- We require the existence of a Ricci-flat cone-metric

$$\mathbf{ds^2(X_8) = dr^2 + r^2 ds^2(Y_7)}$$

so that $\mathbf{Y_7}$ is an Einstein manifold

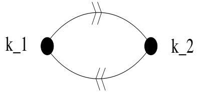
M2-branes at cone singularities

- The "near-horizon" 11d solution is $AdS_4 \times \mathbf{Y_7}$. There are more possibilities for $\mathbf{Y_7}$ now:

$\mathcal{N}$	$\mathbf{Y_7}$	$\mathbf{X_8 = C(Y_7)}$
1	weak $\mathbf{G_2}$	**Spin(7)**
2	Sasaki-Einstein	Calabi-Yau
3	tri-Sasakian	hyper-Kähler
$> \mathbf{3}$	$\mathbf{S^7}/\Gamma$	$\mathbb{C}^4/\Gamma$

M2-branes at cone singularities

- Until 2008 the dual of $AdS_4 \times \mathbf{S^7}$ was not known! ABJM (inspired by BLG) proposed an $\mathcal{N} = \mathbf{6}$ Chern-Simons-matter theory
- It can be written as an $\mathcal{N} = \mathbf{2}$ quiver theory



node $=$ $\mathbf{U(N)}$ CS term at level $\mathbf{k_i}$

$\mathbf{k_1 = -k_2 = k}$ $\mathbf{W}$= polynomial in $\mathbf{X_i}$

Chern-Simons quivers

- $\mathcal{N} = \mathbf{2}$ CS with gauge group $\mathbf{G = U(N_1) \times \cdots \times U(N_n)}$
- Coupled to bi-fundamental "chiral" fields $\mathbf{X_i}$ ("matter")
- Full Lagrangian $\mathcal{L} = \mathcal{L}_{\mathbf{CS}} + \mathcal{L}^{\mathbf{matter}}_{\mathbf{kin}} + \mathbf{W}$
- Relation to 4d $\mathcal{N} = \mathbf{1}$ [more in the second talk]

M2-branes at cone singularities

- $\mathcal{N} = \mathbf{1}$: squashed $\tilde{\mathbf{S}}^7$ is an example. Dual Chern-Simons theory proposed by [Ooguri-Park]. Essentially a less-supersymmetric completion of the ABJM theory
- $\mathcal{N} = \mathbf{3}$: tri-Sasakian metrics abundant. Examples of Chern-Simons quiver duals proposed by [Jafferis-Tomasiello]

Weak $\mathbf{G_2}$ ($\mathcal{N} = \mathbf{1}$) is too hard. Tri-Sasakian ($\mathcal{N} = \mathbf{3}$) is "too easy". The Sasaki-Einstein ($\mathcal{N} = \mathbf{2}$) case is again the most interesting to study

Sasaki-Einstein geometry

- Sasaki-Einstein/related geometry allows to make checks of the AdS/CFT correspondence and predictions in the field theory
- Useful characterizations of a Sasakian manifold $\mathbf{Y}$:
  1. The metric cone $\mathbf{ds^2(X) = dr^2 + r^2 ds^2(Y)}$ is Kähler
  2. Locally the metric can be written as a "fibration"

     $\mathbf{ds^2(Y) = ds^2(B) + (d\psi + P)^2}$ where $\mathbf{B}$ is Kähler

Sasaki-Einstein geometry

- Sasaki-Einstein/related geometry allows to make checks of the AdS/CFT correspondence and predictions in the field theory
- Useful characterizations of a Sasakian manifold **Y**:

  1. The metric cone $\mathbf{ds^2(X) = dr^2 + r^2 ds^2(Y)}$ is Kähler
  2. Locally the metric can be written as a "fibration"

     $\mathbf{ds^2(Y) = ds^2(B) + (d\psi + P)^2}$ where **B** is Kähler

  1. $\frac{\partial}{\partial \psi}$ is a Killing vector ("Reeb") $\Rightarrow$ $\mathbf{U(1)_{R(eeb)}}$ isometry
  2. $\omega = \frac{\mathbf{d}\eta}{2}$, where $\eta = \mathbf{d}\psi + \mathbf{P}$, is the Kähler two-form on **B**

$\mathbf{ds^2(B)}$ is Einstein $\Leftrightarrow$ $\mathbf{ds^2(Y)}$ is Einstein $\Leftrightarrow$ $\mathbf{ds^2(X)}$ is Ricci-flat

Some basic checks of AdS_5/CFT_4

- Isometries $\mathbf{G_{iso}}$ of $\mathbf{Y}$ $\leftrightarrow$ flavour symmetries of field theories
- $\mathbf{U(1)_{R(eeb)}}$ isometry $\leftrightarrow$ $\mathbf{U(1)_R}$ $\mathbf{R}$-symmetry of $\mathcal{N} = \mathbf{1}$ field theories
- If $\mathbf{U(1)_R} \subset \mathbf{U(1)^3} \subset \mathbf{G_{iso}}$, then $\mathbf{Y}$ and $\mathbf{X}$ are toric $\longrightarrow$ great simplifications. Toric Calabi-Yau singularities are characterized by simple combinatorial data, essentially vectors $\mathbf{v_a} \in \mathbb{Z}^3$

Some basic checks of AdS_5/CFT_4

- Isometries G_{iso} of $Y \leftrightarrow$ flavour symmetries of field theories
- $U(1)_{R(eeb)}$ isometry $\leftrightarrow$ $U(1)_R$ R-symmetry of $\mathcal{N} = 1$ field theories
- If $U(1)_R \subset U(1)^3 \subset G_{iso}$, then Y and X are toric $\longrightarrow$ great simplifications. Toric Calabi-Yau singularities are characterized by simple combinatorial data, essentially vectors $v_a \in \mathbb{Z}^3$

$$\langle T^\mu_\mu \rangle = c(\text{Weyl})^2 - a(\text{Euler})$$

- Central charge $a = \dfrac{N^2 \pi^3}{4\,\mathrm{vol}(Y)}$ [Henningson-Skenderis]
- R-charges of certain BPS "baryonic" operators $R_a = \dfrac{N\pi\,\mathrm{vol}(\Sigma_a)}{3\,\mathrm{vol}(Y)}$

  Baryonic operators = D3-branes wrapped on supersymmetric Σ_3

Further checks of AdS_5/CFT_4: matching of moduli spaces

Gauge theory classical moduli spaces of susy vacua (Abelian)

- F-terms: $\mathscr{Z} = \{dW = 0\}$ (a.k.a. "master space")
- D-terms/mod gauge symmetries: $\mathscr{M} = \mathscr{Z}//U(1)^{n-1}$

- $\mathscr{M}$ is the mesonic VMS: gauge-invariant traces $\mathbf{Tr[X_1 \ldots]_{loop}}$
- $\mathscr{Z}$ is the baryonic VMS: determinant-like $\mathbf{det(X_1 \ldots)}$

Further checks of AdS_5/CFT_4: matching of moduli spaces

Gauge theory classical moduli spaces of susy vacua (Abelian)

- F-terms: $\mathscr{Z} = \{\mathrm{dW} = 0\}$ (a.k.a. "master space")
- D-terms/mod gauge symmetries: $\mathscr{M} = \mathscr{Z}//\mathbf{U(1)^{n-1}}$

- $\mathscr{M}$ is the mesonic VMS: gauge-invariant traces $\mathbf{Tr[X_1 \dots]_{loop}}$
- $\mathscr{Z}$ is the baryonic VMS: determinant-like $\mathbf{det(X_1 \dots)}$

Gravity realizations:

- $\mathscr{M}$ is realized simply as $\mathscr{M} = \mathbf{C(Y_5) = X}$. Placing $\mathbf{N}$ D3-branes at generic positions gives $\mathscr{M}_{\mathbf{N>1}} = \mathbf{Sym^N X}$
- Different branches of $\mathscr{Z}$ are realized in the gravity as partial resolutions of the cone singularities $\mathbf{X}$ [Klebanov,Murugan], [DM,Sparks]

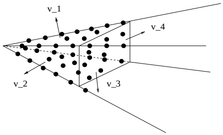
Counting BPS operators

Problem: "count" chiral BPS operators of a quiver theory, labeled by some "quantum number"

- Geometrically, the problem reduces to "counting" holomorphic functions (sections) on the appropriate moduli space
- E.g. on $\mathbb{C}$: $\mathbf{1, z, z^2, z^3, \ldots}$. In general, there are infinitely many holomorphic functions
- Group them into finite sets with definite "quantum numbers". For example $\mathbf{R}$-charges. For toric geometries we can label with $\mathbf{U(1)^3}$ charges $(\mathbf{n_1, n_2, n_3})$
- Counting mesonic BPS operators: enumerate holomorphic functions on $\mathbf{C(Y_5) = X}$ $\longrightarrow$ equivariant index-character on $\mathbf{X}$ [DM,Sparks,Yau]
- Counting baryonic BPS operators: enumerate holomorphic sections on $\mathcal{Z}$. More complicated. [Hanany et al]

Counting BPS operators

- Toric case: holomorphic functions $\leftrightarrow$ integral points inside the cone $\mathcal{C}^*$ (recall $\mathbf{X} \simeq \mathbf{U(1)^3} \rightarrow \mathcal{C}^*$)



$$\mathbf{C(q,X)} = \sum_{\mathbf{n}\in\mathcal{C}^*} \mathbf{q_1^{n_1} q_2^{n_2} q_3^{n_3}}$$

Computed by localization techniques

- Another physical interpretation: the VMS of BPS D3 wrapped in $\mathbf{S^3} \subset \mathrm{AdS}_5$ ("dual-giant gravitons") is $\mathbf{C(Y_5)}$ [DM,Sparks]

- $\mathbf{C(q,X)}$ is the partition function of such states. Grand-canonical partition function

$$\mathcal{Z}(\zeta,\mathbf{q},\mathbf{X}) = \exp\left[\sum_{\mathbf{n}=1}^{\infty} \frac{\zeta^{\mathbf{n}}}{\mathbf{n}} \mathbf{C(q^n,X)}\right] = \sum_{\mathbf{N}=0}^{\infty} \zeta^{\mathbf{N}} \mathbf{Z_N(q,X)}$$

$\mathbf{Z_N}$ counts hol functions on $\mathbf{Sym^N X}$ $\rightarrow$ mesonic BPS operators for $\mathbf{N} > \mathbf{1}$

Volume minimisation and a-maximisation

Slogan: Sasaki-Einstein manifolds minimise volumes [DM,Sparks,Yau]

- More precisely: a Sasakian manifold, as a function of the Reeb vector field, has minimal volume when the metric becomes Einstein
- If the geometry is toric it is easy to visualize: the Reeb $\mathbf{b} \in \mathbb{R}^3$. The volume $\mathrm{vol}(\mathbf{Y})$ of the Sasakian "horizon" $\mathbf{Y}$ as a function of $\mathbf{b}$ is a pole in $\mathbf{C(q, X)}$:

$$\mathrm{vol}(\mathbf{Y})_{\mathbf{b}} = \lim_{t \longrightarrow 0} t^3 C(q_i = e^{-t b_i}, X)$$

- Minimizing $\mathrm{vol}(\mathbf{Y})_{\mathbf{b}}$ gives a $\mathbf{b}_*$, which then can be used to compute the $\mathbf{a}$ central charge and the $\mathbf{R}$-charges of BPS operators

$$a = \frac{N^2 \pi^3}{4\,\mathrm{vol}(Y)_{b_*}} \qquad \Delta_{\mathrm{mesonic}}[n_i] = \sum_i^3 b_*^i n_i$$

Volume minimisation and a-maximisation

- In 4d $\mathcal{N} = 1$ SCFTs this is the geometric counterpart of a-maximisation [Intriligator,Wecht]

$$\langle T^{\mu}_{\mu} \rangle = c(\mathrm{Weyl})^2 - a(\mathrm{Euler}) \qquad a = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R)$$

Introducing a "trial" $R_t = R_0 + \sum_I s^I F_I$; a is maximised over the possible R-symmetries

Volume minimisation and a-maximisation

- In 4d $\mathcal{N} = 1$ SCFTs this is the geometric counterpart of a-maximisation [Intriligator,Wecht]

$$\langle T^{\mu}_{\mu} \rangle = c(\text{Weyl})^2 - a(\text{Euler}) \qquad a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R)$$

  Introducing a "trial" $R_t = R_0 + \sum_I s^I F_I$; a is maximised over the possible R-symmetries

- In 3d SCFTs, the geometry predicts a field theory technique to determine the R-symmetry of $\mathcal{N} = 2$ CS theories

Examples of AdS_5/CFT_4

A complete list of dual pairs where both the Sasaki-Einstein metric and the dual field theory are known explicitly

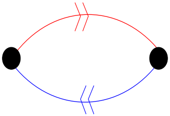
1. $\mathbf{T^{1,1}}$ metric $\longrightarrow$ Klebanov-Witten quiver (1998)
2. $\mathbf{Y^{p,q}}$ metrics [Gauntlett,DM,Sparks,Waldram] $\longrightarrow$ $\mathbf{Y^{p,q}}$ quivers [Benvenuti Franco,Hanany,DM,Sparks] (2004)
3. $\mathbf{L^{a,b,c}}$ metrics [Cvetic,Lu,Page,Pope] $\longrightarrow$ $\mathbf{L^{a,b,c}}$ quivers [several people] (2005)

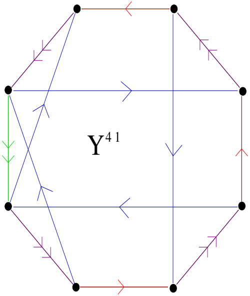
- Lessons from $\mathbf{T^{1,1}}$: first example of non-orbifold AdS/CFT duality; Klebanov-Strassler cascade; and many more.
- Lessons from $\mathbf{Y^{p,q}}$: demonstrated that the volumes of SE manifolds can be irrational multiples of $\mathrm{vol}(\mathbf{S}^5)$. Reflecting the implications of a-maximization

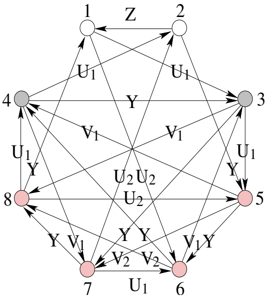
Examples

Klebanov–Witten

(conifold)







AdS_4/CFT_3 correspondence

- Q: What are the fundamental degrees of freedom on M2-branes?

AdS_4/CFT_3 correspondence

- Q: What are the fundamental degrees of freedom on M2-branes?
  A: Despite the recent progress, this is not really clarified
- The lesson of ABJM is that presumably, we should look for Chern-Simons-matter theories
- Sasaki-Einstein results make predictions on the dual $\mathcal{N} = 2$ Chern-Simons theory
- There are a number of proposals for the CFT_3 duals to various AdS_4 geometries
- $\mathcal{N} = 2$ proposals are based on a general result about moduli spaces, which I will discuss in the part II

$\mathcal{N} = 2$ AdS$_4$/CFT$_3$: the regular Sasaki-Einstein manifolds

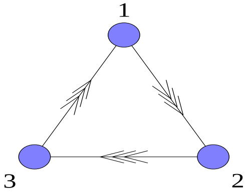
- Before 2004 three known examples of Sasaki-Einstein in **7d** (different generalisations of $\mathbf{T^{1,1}}$):

$$\mathbf{M^{3,2}}, \quad \mathbf{Q^{1,1,1}}, \quad \mathbf{V_{5,2}}$$

- Isometries: $\mathbf{SU(3) \times SU(2) \times U(1)}$, $\mathbf{SU(2)^3 \times U(1)}$, $\mathbf{SO(5) \times U(1)}$
- They are regular i.e. the volumes are rational multiples of $\mathbf{vol(S^7)}$
- In the end-'90s proposals for gauge theory duals were given $\longrightarrow$ problematic; however not Chern-Simons gauge theories
- ABJM wisdom: look at $\mathcal{N} = 2$ Chern-Simons-matter quivers!
- Other ABJM insight: do not attempt to realise all the symmetries in the Lagrangian!

A proposed dual to $AdS_4 \times M^{3,2}/\mathbb{Z}_k$

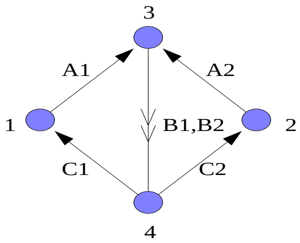
[DM,Sparks]



- The Chern-Simons levels are $(k_1, k_2, k_3) = (k, k, -2k)$
- The superpotential is $W = \epsilon_{ijk} \mathrm{Tr}\,(X_i Y_j Z_k)$
- As a 4d theory it corresponds to the orbifold model $\mathbb{C}^3/\mathbb{Z}_3$
- By construction the moduli space of this CS quiver is $X = C(M^{3,2}/\mathbb{Z}_k)$
- A (partial) check: dimensions of some operators match Kaluza-Klein harmonics on $M^{3,2}/\mathbb{Z}_k$ [Franco,Klebanov,Rodriguez-Gomez]

Proposed duals to $AdS_4 \times \mathbf{Q^{1,1,1}}/\mathbb{Z}_k$

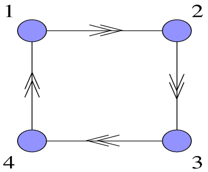
Two different proposed quivers. [Franco,Hanany,Park,Rodriguez-Gomez]



- Chern-Simons levels $(\mathbf{k}, -\mathbf{k}, \mathbf{k}, -\mathbf{k})$.
- The superpotential is $\mathbf{W} = \mathrm{Tr}\,(\mathbf{C_2 B_1 A_1 B_2 C_1 A_2}) - (\mathbf{A_1} \leftrightarrow \mathbf{A_2})$
- It is not well-defined as a 4d theory

Proposed duals to $AdS_4 \times \mathbf{Q}^{1,1,1}/\mathbb{Z}_k$

[Aganagic]



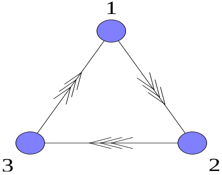
- Chern-Simons levels $(\mathbf{k}, 0, -\mathbf{k}, 0)$
- The superpotential is $\mathbf{W} = \epsilon_{ik}\epsilon_{jl}\mathrm{Tr}\,(\mathbf{A_i B_j C_k D_l})$
- As a 4d theory it corresponds to the an orbifold $\mathbf{T}^{1,1}/\mathbb{Z}_2$
- Both models pass some basic checks: moduli spaces, and matching of some dimensions with Kaluza-Klein spectrum

It is not known if ultimately only one of them is the correct theory; or perhaps the two are related by some duality

$\mathcal{N}=2$ AdS_4/CFT_3: the irregular SE manifolds

- [Gauntlett,DM,Sparks,Waldram]: explicit Sasaki-Einstein metrics $\mathbf{Y^{p,k}(B_{2n})}$ in any $\mathbf{D=2n+3}$ dimension (2004)
- E.g. $\mathbf{Y^{p,k}(\mathbb{C}P^2)}$ is a generalisations of $\mathbf{Y^{p,q}}$ in $\mathbf{d=5}$

  Proposed family of CS quivers [DM,Sparks] has same quiver as $\mathbf{M^{3,2}=Y^{2,3}(\mathbb{C}P^2)}$, but CS levels $\mathbf{(k_1,k_2,k_3)=(2p-k,-p,k-p)}$



- These examples are of "irregular" type: volumes are non rational multiples of $\mathrm{vol}(\mathbf{S^7})$
- Can assign geometric R-charges $\longrightarrow$ irrationals!

$$\mathbf{R_a}=\frac{\pi\,\mathrm{vol}[\Sigma_a]}{6\,\mathrm{vol}(\mathbf{Y_7})} \qquad \Sigma_a \text{ supersymmetric 5-submanifolds}$$

Status of AdS_4/CFT_3 ($\mathcal{N} \geq 2$)

From the explicit examples and the general results we can infer some lessons about AdS_4/CFT_3

1. Supersymmetry not realized manifestly in ABJM [Gustavsson,Rey], [Kwon,Oh,Sohn]
2. Flavour symmetries not manifest either: in the "$\mathbf{k = 1}$" cases we always observe an isometry larger than the symmetries of the proposed Lagrangians
3. In the $\mathcal{N} = 2$ case the conjectured CFTs have generically irrational R-charges! It is currently not known how to compute R-charges in the field theory
4. Volume minimization of Sasaki-Einstein $\mathbf{Y_7}$ strongly suggests a 3d version of a-maximisation

Status of $\mathrm{AdS}_4/\mathrm{CFT}_3$ ($\mathcal{N} \geq 2$)

5. "Counting" of mesonic BPS traces goes through. We can predict the entire BPS Kaluza-Klein spectrum of R-charges
6. Account of non-traces is much more subtle. Monopole operators involved [Benna,Klebanov,Klose]
7. Different duals to a given $\mathrm{AdS}_4 \times \mathbf{Y}_7$ solution. Some are understood as related by 3d mirror symmetry (M-theory lifts), some as 3d Seiberg dualities. There is not yet a clear picture though
8. We still lack an "M-theoretic" understanding of the origin of these Chern-Simons theories

Beyond Sasaki-Einstein: I

Some non-Sasaki-Einstein geometries with interesting AdS/CFT applications

- Warped AdS_5 geometries with non-Freund-Rubin type of fluxes

  1. $AdS_5 \times \mathbf{Y_5}$ in type IIB: e.g. mass-deformations of SCFT (e.g. [Pilch,Warner])
  2. $AdS_5 \times \mathbf{Y_6}$ in M-theory: recently [Gaiotto,Maldacena] identified the field theory duals of $\mathcal{N} = 2$ geometries. There are also several $\mathcal{N} = 1$ explicit solutions [Gauntlett,DM,Sparks,Waldram]!

- Supersymmetry implies existence of $\mathbf{U(1)_R}$. a-maximization implies that these $\mathbf{Y_5}, \mathbf{Y_6}$ manifolds have generically irrational volumes

- Interesting to set up volume minimization for these geometries. Hitchin's "generalized geometry" may be useful [Gabella,Gauntlett,Palti,Sparks,Waldram]

Beyond Sasaki-Einstein: II

- $\mathcal{N} \geq \mathbf{2}$ AdS$_4$ $\times$ $\mathbf{Y_7}$ backgrounds can be reduced to supersymmetric type IIA backgrounds with RR $\mathbf{F_2}$: $[\mathbf{F_2}]$ $\sim$Chern-Simons levels
- If $\mathbf{Y_7}$ is a Sasaki-Einstein manifold, $\mathbf{k_{tot}} = \sum_{\mathbf{nodes}} \mathbf{k_i} = \mathbf{0}$ [DM,Sparks]
- The sum of the CS levels $\mathbf{k_{tot}}$ is proportional to the Romans mass $\mathbf{F_0}$ $\longrightarrow$ supersymmetric AdS$_4$ $\times$ $\mathbf{M_6}$ geometries in massive type IIA [Gaiotto,Tomasiello]
- Explicit massive type IIA solutions
  1. $\mathcal{N} = \mathbf{1}$ deformation of $\mathbf{S^7}$ (ABJM) [Tomasiello]
  2. $\mathcal{N} = \mathbf{2}$ deformation of $\mathbf{M^{3,2}}$ [Petrini,Zaffaroni]
- The field theory analysis suggests a canonical deformation of Sasaki-Einstein solutions. (Recent paper by [Luest,Tsimpis])

Beyond Sasaki-Einstein: III

Fractional branes: the best understood case is the Klebanov-Strassler cascade. Adding fractional branes and deforming the singular conifold geometry leads to a cascade of Seiberg dualities and confinement in the IR

1. In type IIB, deforming many other cones is not possible. Interpreted as runaway behaviour in the 4d $\mathcal{N} = 1$ field theory. Supergravity dual of this not available. Perhaps the perspective in [Maldacena,DM] will be useful

2. In M-theory, fractional M2-branes behave differently. Correspond to torsion fluxes, rather subtle to detect [ABJ]

   Possible to deform some eight-fold singularities, and add fluxes $\longrightarrow$ strong indication of phenomenon analogous to the KS cascade for $\mathcal{N} = 2$ Chern-Simons theories (DM,Sparks WIP). Recent related paper [Aharony,Hashimoto,Hirano,Ouyang]