# The AdS/CFT Correspondence and Sasaki-Einstein Geometry II

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# Topics

1. Non-relativistic backgrounds from massive Kaluza-Klein truncations
2. $AdS_4/CFT_3$ and the $CY_4/CY_3$ connection

# Outline

1. Non-relativistic backgrounds from massive Kaluza-Klein truncations
   - Motivations
   - Non-relativistic conformal symmetries and geometric realization
   - Massive Kaluza-Klein consistent truncations of type IIB supergravity

# Outline

1. Non-relativistic backgrounds from massive Kaluza-Klein truncations
   - Motivations
   - Non-relativistic conformal symmetries and geometric realization
   - Massive Kaluza-Klein consistent truncations of type IIB supergravity
2. $\mathrm{AdS}_4/\mathrm{CFT}_3$ and the $\mathrm{CY}_4/\mathrm{CY}_3$ connection
   - $\mathcal{N} = 2$ Chern-Simons-matter quiver gauge theories
   - Moduli spaces
   - Calabi-Yau four-folds from three-folds and string theory origin of CS theories

# First topic

Non-relativistic backgrounds from massive Kaluza-Klein truncations

# Motivations

- Apply AdS/CFT to (strongly coupled) condensed matter systems
- E.g. "Fermions at unitarity"
- Holography for spaces which are not (asymptotically) anti-de-Sitter
- Non-relativistic limits of string theory

# A physical example: "fermions at unitarity"

**The model (conformal in $\mathbf{d = 2 + \epsilon}$)**

$$\mathbf{S} = \int \mathbf{dtd^d x} \left( \mathbf{i}\psi^\dagger_\alpha \partial_\mathbf{t} \psi_\alpha - \frac{\mathbf{1}}{\mathbf{2m}} (\partial_\mathbf{i} \psi_\alpha)^2 + \mathbf{c}\psi^\dagger_\uparrow \, \psi^\dagger_\downarrow \psi_\uparrow \, \psi_\downarrow \right)$$

Dimensional analysis: $\mathbf{[t] = -2}$, $\mathbf{[x_i] = -1}$, $[\psi_\alpha] = \mathbf{d/2}$, $\mathbf{[c] = 2 - d}$

- quartic interaction irrelevant for $\mathbf{d - 2 > 0}$. RG equation in $\mathbf{d = 2 + \epsilon}$ has two fixed points [Nishida,Son] (UV fixed points: slightly unusual)

  1) $\mathbf{c = 0}$: trivial

  2) $\mathbf{c = 2\pi\epsilon}$: "unitarity" regime, i.e. infinite scattering length

- in $\mathbf{d = 3}$ it is a strongly coupled conformal fixed point $\longrightarrow$ perhaps the AdS/CFT correspondence can be useful?

# Non-relativistic conformal symmetries

## Galilean symmetries

- Generators: time translations **H**; spacial translations $\mathbf{P_i}$; rotations $\mathbf{J_{ij}}$; Galilean boosts $\mathbf{K_i}$
- Non-zero commutators of centrally extended (Bargmann) algebra

$$[\mathbf{H}, \mathbf{K_i}] = -\mathbf{i}\mathbf{P_i} \qquad [\mathbf{P_i}, \mathbf{K_j}] = -\mathbf{i}\delta_{\mathbf{ij}}\mathbf{M} \qquad \text{plus rotations}$$

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- Extension by dilatations $\mathbf{D}$

$$\mathbf{D} : x_i \longrightarrow \lambda x_i \quad t \longrightarrow \lambda^z t \qquad z \text{ "dynamical critical exponent"}$$

$$[\mathbf{D}, \mathbf{P_i}] = -i\mathbf{P_i} \qquad [\mathbf{D}, \mathbf{H}] = -iz\mathbf{H}$$

$$[\mathbf{D}, \mathbf{K_i}] = i(z-1)\mathbf{K_i} \qquad [\mathbf{D}, \mathbf{M}] = i(z-2)\mathbf{M}$$

- Removing the boosts $\mathbf{K_i}$ (and $\mathbf{M}$): $(\mathbf{H}, \mathbf{P_i}, \mathbf{J_{ij}}, \mathbf{D})$ called Lifshitz$_z$ algebra [Kachru,Liu,Mulligan], [Hořava]

# Schrödinger symmetry

- If $z = 2$ consistent to add conformal transformations
- $C : x_i \rightarrow \frac{x_i}{1+at} \quad t \rightarrow \frac{t}{1+at}$ time-dependent expansions
- Additional non-zero commutators. E. g.

$$[D, C] = 2iC \qquad [D, H] = -2iH \qquad [H, C] = iD$$

**In summary**

Galilei $(H, P_i, K_i, J_{ij})$ + central term $M$ = Bargmann

Bargmann + $(D, C)$ = Schrödinger

- Symmetries of the Schrödinger equation

$$2iM\frac{\partial}{\partial t}\Psi + \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\Psi = 0$$

- Other non-relativistic conformal groups exist. [Bagchi,Gopakumar], [DM,Tachikawa]. See talk by Gopakumar

# Geometric realisation of the Schrödinger symmetries

- First evidence of AdS/CFT duality: matching of symmetries on two sides

- $\mathbf{SO(d+1,2)}$ is the (relativistic) conformal group of a $\mathbf{d+1}$ dimensional CFT = isometry group of $\mathrm{AdS}_{\mathbf{d+2}}$

- Are there geometries with Schrödinger symmetry?

- [Son], [Balasubramanian,McGreevy]: Schrödinger group is *embedded* into the relativistic conformal group in two dimensions higher

- $\mathbf{SO(d+2,2)} = \{\tilde{\mathbf{M}}^{\mu\nu}, \tilde{\mathbf{P}}^{\mu}, \tilde{\mathbf{K}}^{\mu}, \tilde{\mathbf{D}}\}$, introduce light-cone coordinates $\mathbf{x}^{\pm} = \mathbf{x}^{0} \pm \mathbf{x}^{d+1}$, $\mathbf{x_i}, \mathbf{i} = 1, \ldots, \mathbf{d}$

$$\mathbf{M} = -\tilde{\mathbf{P}}_{-} \quad \mathbf{H} = -\tilde{\mathbf{P}}_{+} \quad \mathbf{P_i} = \tilde{\mathbf{P}}_{\mathbf{i}} \quad \mathbf{J_{ij}} = \tilde{\mathbf{M}}_{\mathbf{ij}} \quad \mathbf{K_i} = \tilde{\mathbf{M}}_{-\mathbf{i}}$$

$$\mathbf{D} = \tilde{\mathbf{D}} + 2\tilde{\mathbf{M}}_{-+} \qquad \mathbf{C} = -\tilde{\mathbf{K}}_{-}$$

# Geometric realisation of the Schrödinger symmetries

- Embedding into isometries of AdS space hints to a geometric realisation
- $\mathrm{Sch}(\mathbf{d}) \subset \mathbf{SO}(\mathbf{d}+2,2) \quad \longrightarrow$ $\mathrm{Sch}(\mathbf{d})$ metric obtained as a deformation of $\mathrm{AdS}_{\mathbf{d}+3}$

**The Schrödinger invariant metric**

$$\mathbf{ds}^2 = \underbrace{\frac{\mathbf{dr}^2}{\mathbf{r}^2} + \mathbf{r}^2\left[\mathbf{dx_i dx_i} - \mathbf{dx^+ dx^-}\right]}_{\mathbf{AdS_{d+3}}} - \sigma^2 \mathbf{r}^{2\mathbf{z}} (\mathbf{dx}^+)^2$$

- non-relativistic time $\mathbf{t} = \mathbf{x}^+$: $\mathbf{H} = -\partial/\partial \mathbf{x}^+$
- mass (central term): $\mathbf{M} = -\partial/\partial \mathbf{x}^-$
- Schrödinger symmetry requires $\mathbf{z} = \mathbf{2}$. Metrics with $\mathbf{z} \neq \mathbf{2}$ are not invariant under conformal transformations $\mathbf{C}$

# Embedding the metric in string theory

- To build up a holographic dictionary the next step is to see these metrics emerging as solutions of string theory
- The Schrödinger-invariant metric ($z$=2), with $\mathbf{d = 2}$

$$\mathbf{ds^2 = \frac{dr^2}{r^2} + r^2 \left[dx_i dx_i - dx^+ dx^-\right] - \sigma^2 r^4 (dx^+)^2}$$

arises in string theory as a solution of type IIB supergravity

- It could be obtained using a solution generating technique (TsT). [Maldacena,DM,Tachikawa], [Herzog,Rangamani,Ross], [Adams,Balasubramanian,McGreevy]

# Solution to Einstein-Proca equations

- [Son], [Balasubramanian,McGreevy] noticed that the metric

$$\mathbf{ds^2 = -\sigma^2 r^{2z}(dx^+)^2 + \frac{dr^2}{r^2} + r^2\left(-dx^+dx^- + dx_i dx_i\right)}$$

  is as solution of EOMs following from the Einstein-Proca action: gravity coupled to a massive photon

$$\mathbf{S_{EP} = \int d^{d+2}x dr \sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu\right)}$$

- The ansatz for the gauge field is $\mathbf{A_+ \propto r^z}$. Specific relations among the parameters: $\mathbf{\Lambda = -\frac{1}{2}(d+1)(d+2)\ , \quad m^2 = z(z+d)}$

# Kaluza-Klein consistent truncations

- Is $\mathbf{S_{EP}}$ contained in some known gauged supergravity arising as consistent truncations of ten or eleven dimensional supergravities?

# Kaluza-Klein consistent truncations

- Is $\mathbf{S_{EP}}$ contained in some known gauged supergravity arising as consistent truncations of ten or eleven dimensional supergravities?

$$\mathbf{S^{10d}} \longrightarrow \mathbf{10d}\ \text{EOMs} \longrightarrow \mathbf{5d}\ \text{EOMs} \longrightarrow \mathbf{S^{5d}_{sugra}}$$

- The truncation is consistent if any solution to the 5d EOMs can be uplifted to a solution of the 10d EOMs using the truncation ansatz

- Example: 5d minimal gauged sugra is a consistent truncation of type IIB supergravity [Buchel-Liu]. Take a Sasaki-Einstein metric $\mathrm{ds}^2(\mathrm{B_{KE}}) + (\mathrm{d}\psi + \mathbf{P})^2$
$\longrightarrow$ metric ansatz: $\mathrm{ds}^2_{10} = \mathrm{ds}^2_5 + \mathrm{ds}^2(\mathrm{B_{KE}}) + (\mathrm{d}\psi + \mathbf{P} + \mathbf{A})^2$

$$\mathbf{S}_{\text{minimal}} = \int (\mathbf{R} + \Lambda) * 1 - \mathbf{F} \wedge *\mathbf{F} - \mathbf{F} \wedge \mathbf{F} \wedge \mathbf{A}$$

- Special properties of SE structure allow natural ansatz $\longrightarrow$ we can generalize this to massive modes

# Massive truncation I

- A deformation of Sasaki-Einstein $[\eta = \mathrm{d}\psi + \mathbf{P}]$ geometry including: 2 scalars, 1 massive gauge field

$$\mathrm{ds}^2_{10} = \mathrm{e}^{-\frac{2}{3}(4\mathrm{U}+\mathrm{V})}\mathrm{ds}^2(\mathrm{M}_5) + \mathrm{e}^{2\mathrm{U}}\mathrm{ds}^2(\mathrm{B_{KE}}) + \mathrm{e}^{2\mathrm{V}}\eta^2$$

$$\mathrm{B} = \mathrm{A} \wedge \eta\ , \quad \text{dilaton } \phi$$

$$\mathrm{F}_5 = (1+\star)4\mathrm{e}^{-4\mathrm{U}-\mathrm{V}}\,\mathrm{vol}(\mathrm{M}_5)$$

- This ansatz yields a 5d consistent truncation ($\mathbf{u}, \mathbf{v}$ lin combinations of $\mathbf{U}, \mathbf{V}$)

$$\mathrm{S} = \frac{1}{2}\int \mathrm{d}^5\mathrm{x}\sqrt{-\mathrm{g}}\Big[\mathrm{R} + 24\mathrm{e}^{-\mathrm{u}-4\mathrm{v}} - 4\mathrm{e}^{-6\mathrm{u}-4\mathrm{v}} - 8\mathrm{e}^{-10\mathrm{v}} - 5\partial_a\mathrm{u}\partial^a\mathrm{u}$$
$$- \frac{15}{2}\partial_a\mathrm{v}\partial^a\mathrm{v} - \frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{4}\mathrm{e}^{-\phi+4\mathrm{u}+\mathrm{v}}\mathrm{F}_{ab}\mathrm{F}^{ab} - 4\mathrm{e}^{-\phi-2\mathrm{u}-3\mathrm{v}}\mathrm{A}_a\mathrm{A}^a\Big]$$

- $\mathbf{m}_\mathbf{A}^2 = 8 \Rightarrow z = 2\ (d = 2)$ Schrödinger metric is a solution

# Massive truncation II

- The second type IIB ansatz involves only metric and $\mathbf{F_5}$. $(\omega = d\eta/2)$

$$ds^2_{10} = e^{-\frac{2}{3}(4U+V)} ds^2(M_5) + e^{2U} ds^2(B_{KE}) + e^{2V}(\eta + \mathcal{A})^2$$

$$F_5 = (1 + \star_{10}) \left[ 2\omega^2 \wedge (\eta + \mathcal{A} + A) - \omega \wedge (\eta + \mathcal{A}) \wedge \mathbb{F} \right]$$

- $\mathcal{F} = d\mathcal{A}$, $F = dA$, $\mathbb{F} = F + \mathcal{F}$. This ansatz yields a different 5d consistent truncation (below set scalars to zero)

$$S_{vec} = \frac{1}{2} \int d^5x \sqrt{-g} \Big[ -\frac{3}{4} (\mathcal{F} + \frac{2}{3} F)_{ab} (\mathcal{F} + \frac{2}{3} F)^{ab} - \frac{1}{6} F_{ab} F^{ab} - 8 A_a A^a \Big] + S_{CS}$$

- One massless gauge field $\mathcal{A} + \frac{2}{3} A$ and one massive gauge field $\mathbf{A}$ with $m^2_A = 24 \Rightarrow$ metric with dynamical exponent $z = 4$ $(d = 2)$

- It is a massive generalisation of minimal 5D gauged supergravity

# Massive truncation of 11d supergravity

- [Gauntlett,Kim,Varela,Waldram] constructed an analogous massive truncation of eleven dimensional supergravity

$$\mathrm{ds}_{11}^2 = \mathrm{e}^{-\frac{7}{3}\mathbf{v}}\mathrm{ds}^2(\mathbf{M}_4) + \mathrm{e}^{\frac{2}{3}\mathbf{v}}[\mathbf{e}\mathbf{x}^{-2\mathbf{u}}\mathrm{ds}^2(\mathbf{B}_{\mathsf{KE}}) + \mathrm{e}^{12\mathbf{u}}(\eta + \mathcal{A})^2]$$

$$\mathbf{G}_4 = \text{something}$$

- This ansatz yields a 4D consistent truncation
- It is a massive generalisation of minimal 4D gauged supergravity
- It admits a solution with $\mathbf{z} = \mathbf{3}$ and $\mathbf{d} = \mathbf{1}$

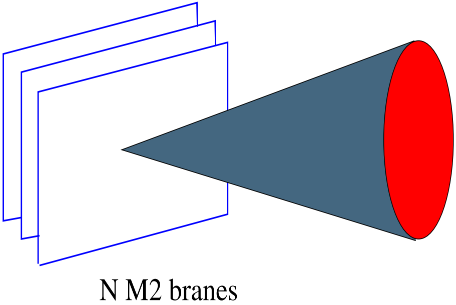
$$\mathrm{ds}^2(\mathbf{M}_4) = -\sigma^2\mathbf{r}^6(\mathrm{dx}^+)^2 + \frac{\mathrm{dr}^2}{\mathbf{r}^2} + \mathbf{r}^2\left(-\mathrm{dx}^+\mathrm{dx}^- + \mathrm{dx}^2\right)$$

# Second topic

$AdS_4/CFT_3$ and the $CY_4/CY_3$ connection

# M2-branes at Calabi-Yau four-fold singularities

- Motivated by ABJM $\longrightarrow$ study $AdS_4/CFT_3$ in $\mathcal{N} \geq 2$ cases
- Place **N M2** branes at a Calabi-Yau four-fold conical singularity $\mathbf{X_8}$



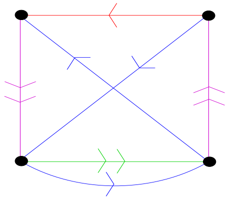

- Existence of a Ricci-flat cone-metric

$$\mathbf{ds^2(X_8) = dr^2 + r^2 ds^2(Y_7)}$$

implies $\mathbf{Y_7}$ is a Sasaki-Einstein seven-manifold

# Chern-Simons quivers

- Understand more systematically $\mathcal{N} = 2$ Chern-Simons quivers




node = $\mathbf{U(N)}$ CS term at level $\mathbf{k_i}$ $\qquad$ **W**= polynomial in $\mathbf{X_i}$

## Chern-Simons quivers

- $\mathcal{N} = 2$ CS with gauge group $\mathbf{G} = \mathbf{U(N_1)} \times \cdots \times \mathbf{U(N_n)}$
- Coupled to bi-fundamental "chiral" fields $\mathbf{X_i}$ ("matter")
- Full Lagrangian $\mathcal{L} = \mathcal{L}_{\mathbf{CS}} + \mathcal{L}^{\mathbf{matter}}_{\mathbf{kin}} + \mathbf{W}$

# $\mathcal{N} = 2$ Chern-Simons Lagrangian

- The general $\mathcal{N} = 2$ Lagrangian is

$$S = S_{\mathrm{CS}} + S_{\mathrm{matter}} + S_{\mathrm{potential}}$$

$$S_{\mathrm{CS}} = \sum_{i=1}^{n} \frac{k_i}{4\pi} \int \mathrm{Tr} \left( A_i \wedge dA_i + \frac{2}{3} A_i \wedge A_i \wedge A_i - \bar{\chi}_i \chi_i + 2 D_i \sigma_i \right)$$

$$S_{\mathrm{matter}} = \int d^3x \sum_a \mathscr{D}_\mu \bar{\phi}_a \mathscr{D}^\mu \phi_a - \bar{\phi}_a \sigma^2 \phi_a + \bar{\phi}_a D \phi_a$$

$$S_{\mathrm{potential}} = -\int d^3x \sum_a \left| \frac{\partial W}{\partial \phi_a} \right|^2$$

- $\mathcal{N} \geq 3$ requires special (quartic) $\mathbf{W}$. We keep it general
- Same Lagrangian for $\mathcal{N} = 1$ quivers in 4D, with $S_{\mathrm{YM}} \to S_{\mathrm{CS}}$

# Moduli spaces

- Consider Abelian theories: $\mathbf{G = U(1)^n}$
- After integrating out the auxiliary fields $\mathbf{D_i}$, the total (bosonic) potential is $\mathcal{V} = \mathcal{V}_D + \mathcal{V}_F$

$$\mathcal{V}_F = \sum_a \left| \frac{\partial W}{\partial \phi_a} \right|^2$$

$$\mathcal{V}_D = \sum_a |\phi_a|^2 (\sigma_{h(a)} - \sigma_{t(a)})^2$$

- In the process we get effective D-terms:

$$- \sum_{a|h(a)=i} |\phi_a|^2 + \sum_{a|t(a)=i} |\phi_a|^2 = \frac{k_i \sigma_i}{2\pi} \quad \forall \quad i$$

- The usual 4d D-terms are LHS $= 0$

# Supersymmetric vacua

- $\mathcal{V}_D, \mathcal{V}_F$ must vanish separately
- F-terms: $$\frac{\partial W}{\partial \phi_a} = 0 \quad \rightarrow \quad \mathcal{Z} = \{dW = 0\} \subset \mathbb{C}^D$$
- D-terms: $$\sigma_1 = \sigma_2 = \cdots = \sigma_n \equiv s$$

$$\mathcal{D}_i = -\sum_{a|h(a)=i} |\phi_a|^2 + \sum_{a|t(a)=i} |\phi_a|^2 = \frac{sk_i}{2\pi} \quad \forall \quad i$$

$$\sum_{i=1}^{n} k_i = 0 \quad \rightarrow \quad n-2 \text{ conditions}$$

# Gauge symmetries

- We should mod by gauge transformations. Naively mod by: $\mathbf{U(1)^{n-1} \cong U(1)^n/U(1)}$. Problematic...
- A particular $\mathbf{U(1)}$ gauge symmetry is broken to a discrete subgroup $\mathbb{Z}_h$ by background monopoles. Defining

$$a = \sum_{i=1}^{n} A_i \qquad b = \frac{1}{h}\sum_{i=1}^{n} k_i A_i \qquad h = \mathrm{hcf}(k_i)$$

the (Abelian) CS action is

$$S_{\mathrm{CS}}(A_i) = \frac{h}{2\pi n}\int b \wedge da + S'$$

where under $A_i \longrightarrow A_i + \lambda$, $\delta S' = 0$. Then we can dualize $\mathbf{b}$ to a periodic scalar $\tau$ integrating out $\mathbf{f}$

$$b = \frac{n}{h} d\tau \qquad \tau \in [0, \frac{2\pi}{n}] \quad (*)$$

- Gauge transformations:

$$\mathsf{A_i} \longrightarrow \mathsf{A_i} + d\theta_\mathsf{i}\ , \qquad \sum_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{k_i}\theta_\mathsf{i} = 0\ ; \qquad \tau \longrightarrow \tau + \frac{\mathsf{h}}{\mathsf{n}}\theta$$

Thus $\theta = \frac{2\pi \mathsf{l}}{\mathsf{h}}$ $\mathsf{l} = 1, \ldots, \mathsf{h}$ is a residual gauge symmetry.

- May be summarized as "Kernel of the character":

$$\chi_\mathsf{k} : \mathsf{U(1)^n} \longrightarrow \mathsf{U(1)}$$

$$\left(e^{\mathsf{i}\theta_1}, \ldots, e^{\mathsf{i}\theta_\mathsf{n}}\right) \longmapsto \exp\left(\mathsf{i}\sum_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{k_i}\theta_\mathsf{i}\right)$$

- The gauge symmetries are then the group

$$\mathsf{H_k} = \ker \chi_\mathsf{k}/\mathsf{U(1)} \cong \mathsf{U(1)^{n-2}} \times \mathbb{Z}_\mathsf{h}$$

- The $\mathsf{U(1)^{n-2}}$ part is the same group for which we imposed D-terms

$$\sum_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{v_i}\mathcal{D}_\mathsf{i} = 0, \qquad \mathsf{v} \in \ker(\mathsf{k})$$

# Periodicity of $\tau$

- Justification of $\tau \in [0, \frac{2\pi}{n}]$

$$f = da \qquad b = \frac{1}{h}\sum_{i=1}^{n} k_i A_i \qquad S_{CS}(A_i) = \frac{h}{2\pi n}\int b \wedge f + S'$$

- $b$ is dualized adding $S_\tau$

$$S_\tau = -\frac{1}{2\pi}\int d\tau \wedge f \quad \Rightarrow \quad b = \frac{n}{h} d\tau$$

- Periodicity fixed by $\int_{\sigma_2} f \neq 0$
  - $\int_{\sigma_2} F_i \in 2\pi\mathbb{Z}$
  - $\sum_{a\in\mathcal{A}} |\phi_a|^2 (A_{h(a)} - A_{t(a)})^2 = 0 \quad \Rightarrow \quad F_1 = \cdots = F_n$

$$\int_{\sigma_2} f \in 2\pi n\,\mathbb{Z}$$

# Connection to Calabi-Yau three-fold

- The moduli space contains a $\mathbf{4}_{\mathbb{C}}$-dimensional branch

$$\mathscr{M}_{\mathbf{3d}}(\mathbf{k}) = \mathscr{Z}//\mathbf{H_k}$$

- If we quotiented by the $\mathbf{U(1)}$ symmetry broken by monopoles we would obtain a $\mathbf{3}_{\mathbb{C}}$-dimensional space

$$\mathscr{M}_{\mathbf{4d}} \equiv \mathscr{M}_{\mathbf{3d}}(\mathbf{k})//\mathbf{U(1)}$$

- If we start from a theory with a "parent" 4d $\mathcal{N} = 1$ quiver, then $\mathscr{Z}$ is the baryonic moduli space of the 4d quiver theory

## 3/4 connection

- $\mathscr{M}_{\mathbf{4d}}$ Calabi-Yau 3-fold $\Rightarrow$ $\mathscr{M}_{\mathbf{3d}}(\mathbf{k})$ Calabi-Yau 4-fold
- M-theory 4-fold is a fibration over the 3-fold: $\mathscr{M}_{\mathbf{3d}}(\mathbf{k}) \longrightarrow \mathbb{C}^* \longrightarrow \mathscr{M}_{\mathbf{4d}}$

# String theory origin of the Chern-Simons theories

[Aganagic]

- From these geometric results one can infer a string theory origin of the Chern-Simons theories
- Take $\mathbf{N}$ D2-branes at a Calabi-Yau three-fold $\mathbf{X_3}$ singularity$\times\mathbb{R}$
- T-dual to D3 at $\mathbf{X_3}$: gauge theory on these is simply the dimensional reduction of a 4d $\mathcal{N} = \mathbf{1}$ quiver $\longrightarrow$ 3d $\mathcal{N} = \mathbf{2}$ Yang-Mills quiver
- Add fractional branes = D4 branes wrapped on vanishing $\mathbf{C_i} \subset \mathbf{X_3}$, and turn on RR fluxes

$$\mathbf{S^{WZ}_{D4}} \sim \int_{\mathbb{R}^{1,2}\times \mathbf{C_i}} \mathbf{A} \wedge \mathbf{dA} \wedge \mathbf{F^{RR}_2} = \int_{\mathbf{C_i}} \mathbf{F^{RR}_2} \cdot \mathbf{S_{CS}}$$

$\longrightarrow$ Chern-Simons terms are induced in the world-volume theories

# String theory origin of the Chern-Simons theories

- Uplift to M-theory: at strong coupling $\mathcal{L}_{YM} \to 0 \Rightarrow$ CSM theory
- To compute the CS levels consider the M-theory $\mathbf{U(1)}$ fibration

$$\mathbf{X_4 \to U(1) \to [X_3 \times \mathbb{R}]}$$

$$\Rightarrow \quad \mathbf{[F_{RR}] = \sum_i q_i[\omega_i]}$$

- Every node corresponds to a particular fractional brane $\sim \mathbf{[C_i]}$

$$\mathbf{k_i = \int_{C_i} F_{RR} = \sum_j q_j \cdot \int_{C_i} \omega_j}$$

# Example: string theory origin of ABJM

- For illustration consider the ABJM model $\mathbf{G = U(1)_k \times U(1)_{-k}}$
- The F-terms are trivial: $\mathscr{Z} = \mathbb{C}^4$
- The only possible non-trivial $\mathbf{U(1) = U(1)_{rel}}$ is broken to the sub-group $\mathbf{H_K} = \mathbb{Z}_\mathbf{k}$. Thus $\mathscr{M}_{\mathbf{3d}}(\mathbf{k}) = \mathbb{C}^4/\mathbb{Z}_\mathbf{k}$
- The would-be quotient by $\mathbf{U(1)_{rel}}$ is the Kähler quotient of $\mathbb{C}^4$ by

$$|\mathbf{a_1}|^2 + |\mathbf{a_2}|^2 - |\mathbf{b_1}|^2 - |\mathbf{b_2}|^2 = \mathbf{t}$$

- The Chern-Simons level for the two nodes are computed using

$$[\mathbf{F_{RR}}] = \mathbf{k}[\omega]_{\text{resolved conifold}}$$

$$\mathbf{k_1} = -\mathbf{k_2} = \mathbf{k}\int_{\mathbb{C}\mathbf{P}^1} \omega_{\text{resolved conifold}} = -\mathbf{k}$$

# Summary

1. In the first part
   - Non-relativistic conformal (Schrödinger) symmetry
   - New consistent truncation of type IIB on Sasaki-Einstein manifolds
   - Expect these massive consistent truncations to have several other applications. E.g. after appropriately supersymmetrized $\longrightarrow$ find several new susy solutions of type IIB sugra
2. In the second part
   - A closer look at $\mathcal{N} = \mathbf{2}$ Chern-Simons-matter quivers
   - Geometry of moduli spaces $\longrightarrow$ string theory origin of these theories
   - Useful conceptually and practically. Gives a method to derive a Chern-Simons quiver from a given $\mathrm{AdS}_4 \times \mathbf{Y_7}$ M-theory solution
   - Application to study cascading Chern-Simons theories (WIP)