Collocation – an efficient tool for solving singular ODEs and DAEs

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During recent years, a lot of scientific work concentrated on the analysis and numerical treatment of boundary value problems (BVP) in ordinary differential equations (ODEs) which can exhibit singularities. Such problems have often the following form:

$$y'(t) = \frac{1}{t}M(t)y(t) + f(t, y(t)), \quad t \in (0, 1], \quad b(y(0), y(1)) = 0,$$

where f is assumed to be appropriately smooth. Here, we will focus on the slightly more difficult situation, where the functions f is unsmooth, $f(t, y(t)) = g(t, y(t))/t$, g smooth. We first deal with the analytical properties of the problem, existence and uniqueness of smooth solutions, especially with the most general boundary conditions which guarantee its well-posedness.

The search for efficient numerical methods to solve the above BVP is strongly motivated by numerous applications from physics, chemistry, mechanics, ecology, or economy. In particular, problems posed on semi-infinite intervals, $[0, \infty)$ are frequently transformed to a finite domain taking above form with the leading term $M(t)y(t)/t^\alpha$, $\alpha > 1$. Also, research activities in related fields, like differential algebraic equations (DAEs) or singular Sturm-Liouville eigenvalue problems benefit from techniques developed for singular BVPs.

The method of choice for the numerical solution of the singular ODEs is polynomial collocation which is robust with respect to the singularity and retains its advantageous convergence properties known for regular ODEs. For collocation at equidistant points or Gaussian points this convergence results mean that the scheme with m inner collocation points constitutes a high order basic solver whose global error is $O(h^m)$ uniformly in t [5]. Due to the robustness of collocation, this method was used in one of the best established standard FORTRAN codes for (regular) BVPs, COLSYS, as well as in Matlab codes bvp4c, the standard module for (regular) ODEs with an option for singular problems, BVP SOLVER, sbvp [1], and bvpsuite [6]. The scope of bvpsuite includes fully implicit form of the ODE system with multi-point boundary conditions, arbitrary mixed order of the differential equations including zero, module for dealing with infinite intervals, module for eigenvalue problems, free parameters, and a path-following strategy for parameter-dependent problems with turning points. We will illustrate how bvpsuite can be used to solve BVPs from applications [2].

Finally, we turn to DAEs. Especially, higher index DAEs constitute a really challenging class of problems due to the differentiation involved in the solution process which is a critical operation to carry out numerically. A possible technique to master the problem is to pre-handle the DAE system in such a way that the transformed problem is of index 1 and less difficult to solve. Since this approach is technically involved, it is worth to try to avoid it and provide a method which can be applied directly to the original DAE system of high index. We will first discuss the easier index-1 case for which analytical results are already available and then illustrate the performance of the least squares collocation for higher index DAEs [3, 4].

References

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