Holonomy Group of Pseudo-Riemannian Cones

Joint work with
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Let (M,g) be a pseudo-Riemannian manifold of signature $(p,q) = (-,\cdots,-,+,\cdots,+)$. We denote by H the **holonomy group** of (M,g) and by $\mathfrak{h} \subset \mathfrak{so}(V)$ $(V = T_pM$, $p \in M)$ the corresponding **holonomy algebra.**
We say that
$\mathfrak{h}$ is **decomposable**
if V contains a proper non-degenerate $\mathfrak{h}$-invariant subspace.

By Wu's theorem this means that M is locally decomposed as a product of two pseudo-Riemannian manifolds.

In the opposite case $\mathfrak{h}$ is called **indecomposable**.

We say that $\mathfrak{h}$ is **reducible** if it preserves a (possibly degenerate) proper subspace of V.

The holonomy algebra $\mathfrak{h}$ is of exactly one of the following types:

(i) decomposable,

(ii) reducible indecomposable or

(iii) irreducible.

We say that the reducible indecomposable holonomy algebra $\mathfrak{h}$ is of **para-Kähler type** if it preserves a non trivial direct sum decomposition $V = V_1 + V_2$ of V. This means that the manifold (M, g) has a para-Kähler structure that is a parallel field J of skew-symmetric endomorphisms with $J^2 = \mathrm{Id}$ (L. Bererd-Bergery, M. Olbricht).

Let

$$(C_\varepsilon M = \mathbb{R}^+ \times M, \widehat{g} = \widehat{g}_\varepsilon = \varepsilon dr^2 + r^2 g)$$

be the space-like for $\varepsilon = 1$ or timelike for $\varepsilon = -1$ metric cone over (M, g). We denote by

$$\widehat{H} = \widehat{H}_\varepsilon \subset SO(\widehat{V}),\, \widehat{V} = T_p M,\, p \in M$$

the holonomy group of $(\widehat{M}, \widehat{g}_\varepsilon)$ and by

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}_\varepsilon} \subset \mathfrak{so}(\widehat{V})$$

the corresponding holonomy algebra.
It is sufficient to consider the spacelike cone $\widehat{M} := C_+(M)$. We study when the holonomy algebra of the cone $\widehat{M}$ is of one of the types (i-iii).

Theorem 1 (S. Gallot, 1979) *Let* (M, g) *be a complete Riemannian manifold. If the Riemannian cone* $\widehat{M}$ *has decomposable holonomy algebra* $\widehat{\mathfrak{h}}$ *then* (M, g) *has constant curvature 1 and the cone is flat. If, in addition,* (M, g) *is simply connected, then it is equal to the standard sphere.*

Applications

1.[Ch.Bär] **Classification of Riemanian manifolds which admits real Killing spinors.**

2. **In pseudo-Riemannian case, an application for construction of Einstein pseudo-Riemannian and Riemannian metrics in a given conformal class.**

Conformal pseudo-Riemannian structure $c = [g]$ on a manifold M defines the Tractor bundle $T \to M$ with Tractor connection ∇^T. Parallel sections of the tractor bundle T correspond to Einstein metrics in the conformal class c and, hence, are determined by the holonomy group H of ∇^T.

If the conformal class c contains an Einstein metric then H can be identified with the holonomy group of the Levi-Civita connection of Feffermann metric, associated with c. Moreover, this metric is the cone metric over a pseudo-Riemannian manifold.

Case i). The cone $\widehat{M}$ over a pseudo-Riemannian manifold M has decomposable holonomy algebra $\hat{h}$

Local result

Theorem. Let (M, g) be a pseudo-Riemannian manifold with decomposable holonomy algebra $\hat{h}$ of the cone $\widehat{M}$. Then the manifold (M, g) has full holonomy algebra $so(p, q)$, where (p, q) is the signature of the metric g. Furthermore, there exists an open dense submanifold $M' \subset M$ such that any point $p \in M'$ has a neighborhood isometric to a pseudo-Riemannian manifold of the form

$$(a, b) \times N_1 \times N_2$$

with the metric given either by

$$g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2$$

or $g=-ds^2+\cosh^2(s)g_1+\sinh^2(s)g_2$,
where g_1 and g_2 are metrics on N_1 and N_2 respectively.

The case when (M,g) is complete

Theorem 2 *Let (M,g) be a complete pseudo-Riemannian manifold with decomposable holonomy algebra $\widehat{h}$ of the cone $\widehat{M}$. Then there exists an open dense submanifold $M'\subset M$ such that any point $p\in M'$ has a neighborhood $M(p)$ which either*
1) has constant curvature 1 or
2) is isometric to the double warped product

$$N=\mathbb{R}^+\times N_1\times N_2(-1),$$

$$g=-ds^2+\cosh^2(s)g_1+\sinh^2(s)g_2,$$

where $(N_1,g_1),(N_2(-1),g_2)$ are pseudo-Riemannian manifolds and $(N_2(-1),g_2)$ has constant curvature -1.

Gallot Theorem for compact, complete pseudo-Riemannian manifold

Theorem 3 *Let (M, g) be a compact complete pseudo-Riemannian manifold with decomposable holonomy algebra $\hat{h}$ of the cone $\hat{M}$. Then (M, g) has constant curvature 1 and the cone is flat.*

Since there is no simply connected compact indefinite pseudo-Riemannian manifold of constant curvature 1, we get

Corollary 1 *If (M, g) is a simply connected compact and complete indefinite pseudo-Riemannian manifold, then the holonomy algebra of the cone $(\widehat{M}, \widehat{g})$ is indecomposable.*

Case (ii.a) when the holonomy algebra $\widehat{\mathfrak{h}}$ of the cone $\widehat{M}$ is of para-Kähler type

Definition 1 *A para-Sasakian structure on a pseudo-Riemannian manifold (M, g) of signature $(n+1, n)$ is a timelike unit Killing vector field T such that $J = \nabla T|_{T^\perp}$ is an integrable para-complex structure on the (contact) distribution $T^\perp$.*

Theorem 4 *Let (M, g) be a pseudo-Riemannian manifold. There is one-to-one correspondence between para-Sasakian structures (M, g, T) on (M, g) and para-Kähler structures $(\widehat{M}, \widehat{g}, \widehat{J})$ on the cone $(\widehat{M}, \widehat{g})$. The correspondence is given by*
$T \mapsto \widehat{J} := \widehat{\nabla} T$.

Similarly, we have the following characterization of the case when the cone $\widehat{M}$ over a $4n$-dimensional manifold M admits (locally) a para-hyper-Kähler structure, that is three parallel skew-symmetric anticommuting endomorphisms $J_1, J_2, J_3 = J_1 J_2$ such that $J_1^2 = J_2^2 = \mathrm{Id}$. This means that the holonomy algebra $\widehat{\mathfrak{h}} \subset sp(n, \mathbb{R})$.

Definition 2 *A 3 para-Sasakian structure on a pseudo-Riemannian manifold (M, g) is a triple (T_1, T_2, T_3) of unite Killing vector fields such that T_1, T_2 are para-Sasakian structures, $[T_i, T_j] = T_k$ and $\nabla T_3|_{T_3^\perp}$ is an integrable complex structure on the contact distribution $T_3^\perp$.*

Theorem 5 *Let (M, g) be a $(4n-1)$-dimensional pseudo-Riemannian manifold. There is a natural 1-1 correspondence between 3 para-Sasakian structures (T_1, T_2, T_3) on M and para-hyper-Kähler structures $(\hat{J}_1, \hat{J}_2, \hat{J}_3)$ on the cone $(\hat{M}, \hat{g})$, given by*

$$T_i \mapsto \hat{J}_i := \hat{\nabla} T_i, \ i = 1, 2, 3.$$

Case (ii.b) when $\widehat{M}$ is the Lorentzian cone with indecomposable reducible holonomy algebra $\widehat{\mathfrak{h}}$.

Theorem 6 *Let (M, g) be a Lorentzian manifold of signature $(1, n-1)$ or a negative definite Riemannian manifold and $(\widehat{M} = \mathbb{R}^{+} \times M, \widehat{g})$ the cone over M with Lorentzian signature $(1, n)$ or $(n, 1)$ respectively. If the holonomy algebra $\widehat{\mathfrak{h}}$ of $\widehat{M}$ is indecomposable reducible (i.e. preserves an isotropic line $\mathbb{R}p$) then it preserves a non-zero isotropic vector p that is*
$\widehat{\mathfrak{h}} \subset p \wedge E$ *where*
$V = \mathbb{R}p + \mathbb{R}q + E$, $p^2 = q^2 = 0, < p, q >= 1,$
$< p, E >=< q, E >= 0.$

The next theorem treats the case of a Lorentzian cone $\widehat{M} = C_+(M)$ over a negative definite Riemannian manifold M.

Theorem 7 *Let (M, g) be a negative definite Riemannian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric of signature $(+, -, \cdots, -)$. If $\widehat{M}$ admits a non-zero parallel isotropic vector field then M is isometric to a manifold of the form*

$$(M_0 = (a, b) \times N, g = -ds^2 + e^{-2s} g_N), \qquad (1)$$

where $a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ a < b$ and (N, g_N) is a negative definite Riemannian manifold. Furthermore, if $\widehat{\mathfrak{h}}$ is indecomposable then

$$\widehat{\mathfrak{h}} \cong \mathfrak{hol}(N, g_N) \times \mathbb{R}^{\dim N}.$$

The manifold (M, g) is complete if and only if (N, g_N) is complete and $(a, b) = \mathbb{R}$.

The next theorem treats the case of a Lorentzian cone $\widehat{M} = C_+(M)$ over a Lorentzian manifold M of signature $(-, +, \cdots, +)$.

Theorem 8 *Let (M, g) be a Lorentzian manifold and $(\widehat{M}, \widehat{g})$ the cone over M equipped with the Lorentzian metric. If $\widehat{M}$ admits a non-zero parallel isotropic vector field then there exists an open dense submanifold $M' \subset M$ such that any point $p \in M'$ has a neighborhood isometric to a manifold of the form (1), where (N, g_N) is a positive definite Riemannian manifold. Furthermore, if $\mathfrak{hol}(\widehat{M_0})$ is indecomposable then*

$$\mathfrak{hol}(\widehat{M_0}, \widehat{g}) \cong \mathfrak{hol}(N, g_N) \times \mathbb{R}^{\dim N}.$$

If the manifold (M, g) is complete, then we may assume that $(a, b) = \mathbb{R}$.

Preliminaries and examples

Let $\widehat{g} = cdr^2 + r^2 g$ be the cone metric on $\widehat{M} := \mathbb{R}^+ \times M$, where (M, g) is a pseudo-Riemannian manifold. Depending on the sign of the constant c the cone is called space-like ($c > 0$) or time-like ($c < 0$). Of course, later on we will assume that $c = 1$ as we allow g to be negative definite.

We denote by ∂_r the radial vector field. The Levi-Civita connection of the cone $(\widehat{M}, \widehat{g})$ is given by

$$\left.\begin{array}{rcl} \widehat{\nabla}_{\partial_r}\partial_r & = & 0, \\ \widehat{\nabla}_X \partial_r & = & \frac{1}{r}X, \\ \widehat{\nabla}_X Y & = & \nabla_X Y - \frac{r}{c} g(X,Y)\partial_r, \end{array}\right\} \qquad (2)$$

for all vector fields $X, Y \in \Gamma(T\widehat{M})$ orthogonal to ∂_r. Using this we can calculate the curvature $\hat{\mathcal{R}}$ of the cone, which is given by the following formulas including the curvature $\mathcal{R}$ of the base metric g:

for $X, Y, Z,\ U \in TM$. This implies that if (M, g) is a space of constant curvature κ, i.e.

$$\mathcal{R}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{U}) = \kappa \left(\}(\mathcal{X}, \mathcal{U}) \}(\mathcal{Y}, \mathcal{Z}) - \}(\mathcal{X}, \mathcal{Z}) \}(\mathcal{Y}, \mathcal{U}) \right), \tag{3}$$

the cone has the curvature $r^2 \left(\kappa - \frac{1}{c} \right) (g(X, U) g(Y, Z)$
In particular, if $\kappa = \frac{1}{c}$, then the cone is flat, as it is the case for the $c = 1$ cone over the standard sphere of radius 1 or the $c = -1$ cone over the hyperbolic space.

Since the metric $\widehat{g}^- := -dr^2 + r^2 g$ of the time-like cone $\widehat{M}^-$ over (M, g) is obtained by multiplying the metric $dr^2 - r^2 g$ of the space-like cone over $(M, -g)$ by -1 it is sufficient to consider only space-like cones. Thus from now on we assume that $c = 1$.

We will now present some examples which illustrate that Gallot's statement is not true in arbitrary signature, and that the assumption of completeness is essential even in the Riemannian situation.

Theorem 9 *Let (F, g_F) be a complete pseudo-Riemannian manifold. Then the pseudo-Riemannian manifold*

$$(M = \mathbb{R} \times F, g = -ds^2 + \cosh^2(s) g_F)$$

is complete, the restricted holonomy group of the cone over (M, g) is non-trivial and admits a non-degenerate invariant proper subspace.

The fact that the manifold (M, g) is complete can be checked using O'Neills formulas for warped products as in the case of Riemannian F, cf. [**?**, Cor. 25]. The non-vanishing terms of the Levi-Civita connection ∇ of (M, g) are given by

$$\left.\begin{array}{rcl} \nabla_X \partial_s &=& \tanh(s) X, \\ \nabla_{\partial_s} X &=& \partial_s X + \tanh(s) X, \\ \nabla_X Y &=& \nabla^F_X Y + \cosh(s)\sinh(s) g_F(X, Y)\partial_s, \end{array}\right\} \tag{4}$$

where $X, Y \in TF$ are vector fields depending on the parameter s and ∇^F is the Levi-Civita connection of the manifold (F, g_F). Consider on $\widehat{M}$ the vector field $X_1 = \cosh^2(s)\partial_r - \frac{1}{r}\sinh(s)\cosh(s)\partial_s$. We have $\widehat{g}(X_1, X_1) = \cosh^2(s) > 0$. It is easy to check that the distribution generated by the vector field X_1 and by the distribution $TF \subset T\widehat{M}$ is parallel.

For the curvature tensor $\mathcal{R}$ of (M, g) we have

$$\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Z} \;=\; \mathcal{R}_{\mathcal{F}}(\mathcal{X},\mathcal{Y})\mathcal{Z} + \sinh^{\in}(\int)\,(\}_{\mathcal{F}}(\mathcal{Y},\mathcal{Z})\mathcal{X} - \tag{5}$$

where $X, Y, Z,\ U \in TF$ and $\mathcal{R}_{\mathcal{F}}$ is the curvature tensor of (F, g_F). This shows that (M, g) cannot have constant sectional curvature. Thus the cone $(\widehat{M}, \widehat{g})$ is not flat.

Let M be a manifold of the form $\mathbb{R} \times N$ with the metric $g = -(dt^2 + e^{-2t} g_N)$, where (N, g_N) is a pseudo-Riemannian manifold. Then

1. The light-like vector field $e^{-t}(\partial_r + \frac{1}{r}\partial_t)$ on the space-like cone $\widehat{M}$ is parallel.

2. If $(N = N_1 \times N_2, g_N = g_1 + g_2)$ is a product of a flat pseudo-Riemannian manifold (N_1, g_1) and of a non-flat pseudo-Riemannian manifold (N_2, g_2), then $\widehat{M}$ is locally decomposable, not flat and there is a parallel non-degenerate flat distribution on $\widehat{M}$.

Note that the manifold (M, g) in Example **??** is complete if and only g_N is complete and positive definite. Notice that g is the hyperbolic metric in horospherical coordinates if (N, g_N) is Euclidian space.

Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds. Then the product of the cones

$(\widehat{M_1} \times \widehat{M_2} = (\mathbb{R}^+ \times M_1) \times (\mathbb{R}^+ \times M_2), \hat{g} = (dr_1^2 + r_1^2 g_1) + ($

is the space-like cone over the manifold

$$\left(M = \left(0, \frac{\pi}{2}\right) \times M_1 \times M_2, g = ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2\right.$$

Consider the functions

$$r = \sqrt{r_1^2 + r_2^2} \in \mathbb{R}^+, \quad s = \operatorname{arctg}\left(\frac{r_2}{r_1}\right) \in \left(0, \frac{\pi}{2}\right).$$

Since $r_1, r_2 > 0$, the functions r and s give a diffeomorphism $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \times \left(0, \frac{\pi}{2}\right)$. For $\widehat{M_1} \times \widehat{M_2}$ we get

$$\widehat{M_1} \times \widehat{M_2} \cong \mathbb{R}^+ \times \left(0, \frac{\pi}{2}\right) \times M_1 \times M_2$$

and

$$\widehat{g_1} + \widehat{g_2} = dr^2 + r^2(ds^2 + \cos^2(s) g_1 + \sin^2(s) g_2).$$

Suppose that the manifolds (M_1, g_1) and (M_2, g_2) are Riemannian, then the manifold (M, g) is Riemannian and it is not complete. The cone over M is decomposable and non-flat. Example shows that the completeness assumption in Theorems **??** and **??** is necessary.

Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds. Then the product of the truncated cones

$(\widehat{M_1} \times \widehat{M_2}^- = ((1, +\infty) \times M_1) \times ((0,1) \times M_2), \widehat{g_1} + \widehat{g_2}^- =$

is the space-like cone over the manifold

$(M = \mathbb{R}^+ \times M_1 \times M_2, g = -ds^2 + \cosh^2(s) g_1 + \sinh^2(s)$

Consider the functions

$$r = \sqrt{r_1^2 - r_2^2} \in \mathbb{R}^+, \quad s = \operatorname{arcth}\left(\frac{r_2}{r_1}\right) \in \mathbb{R}^+.$$

The functions r and s give a diffeomorphism $(1, +\infty) \times (0,1) \to \mathbb{R}^+ \times \mathbb{R}^+$. For $\widehat{M_1} \times \widehat{M_2}^-$ we get

$$\widehat{M_1} \times \widehat{M_2}^- \cong \mathbb{R}^+ \times \mathbb{R}^+ \times M_1 \times M_2$$

and

$$\widehat{g_1}+\widehat{g_2}^- = dr^2+r^2(-ds^2+\cosh^2(s)g_1+\sinh^2(s)g_2).$$

Let $(t, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ be coordinates on $\mathbb{R}^{2n+1}$ and consider the metric g given by

$$g = \left(\begin{array}{c|c|c} -1 & 0 & u^t \\ \hline 0 & 0 & H^t \\ \hline u & H & G \end{array}\right) \tag{6}$$

in which

- $u = (u_1, \ldots, u_n)$ is a diffeomorphism of $\mathbb{R}^n$, depending on $x_1, \ldots, x_n$,

- $H = 1/2\left(\partial(u_i)/\partial x_j\right)_{i,j=1}^n$ its non-degenerate Jacobian, and

- G the symmetric matrix given by $G_{ij} = -u_i u_j$.

Then the space-like cone over $(\mathbb{R}^{2n+1}, g)$ is not flat but its holonomy representation decomposes into two totally isotropic invariant subspaces. For the proof of this see Proposition **??** in Section **??**.