Quantum geometry and quantum gravity

J. Fernando Barbero G.
Instituto de Estructura de la Materia, CSIC.
XVI Fall Workshop on Geometry and Physics
Lisboa, September 5-8, 2007

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

Session 2: Quantization with the new variables

1. General aspects of quantization.
2. Quantum configuration space for field theories. A simple example: The scalar field.
3. Quantum configuration space for connection field theories.
4. The Hilbert space.
5. The Ashtekar-Lewandowski measure.
6. A useful orthonormal basis.

QUANTIZATION (Dirac quantization for constrained systems)

- Choose a Poisson $*$-algebra of **elementary classical variables** (family of functions in phase space that separate points; in classical mechanics one usually takes q and p).
- Obtain a **representation** of the algebra in a *kinematical* Hilbert space $\mathcal{H}_{kin}$ of "quantum states"
  1. Find a suitable vector space of quantum states states.
  2. Find an inner product. (This will lead us to ask ourselves for the relevant measures).
  3. Find physically interesting orthonormal bases where some important operators take simple forms (for example are diagonal).
- Find self-adjoint operators **representing the constraints** (defined in $\mathcal{H}_{kin}$ or in the dual of a certain dense subspace of it) or the elements of the group of symmetry transformations. Find in this way an appropriate physical Hilbert space $\mathcal{H}_{phys}$ of "quantum states".

- **Extract physics.** Though this is expressed in rather vague terms this is a very important part of the whole business. It requires a lot of work both on the physical and mathematical sides.
  - Find a (complete in some sense) set of self-adjoint operators in the physical Hilbert space representing **observables**. Among them one should select the best suited for experimental or observational measurements. [concrete predictions with observable (astrophysics, cosmology) or experimental (**?**) consequences must be made!].
  - It also requires us to understand the **classical limit** (i.e. how we recover the macroscopic space-time geometry form the quantum model). This is straightforward for free theories (for example pure EM) but highly non-trivial for interacting theories. Notice that even for simple quantum mechanical systems such as the hydrogen atom this is highly non-trivial (no coherent states are known in this case!).
  - Almost certainly a successful implementation of this program will require the development of some sort of approximation scheme (a new *perturbation theory*!).

I will concentrate on the construction of the kinematical Hilbert space

- An example to put things in perspective: **The free scalar field in Minkowski space-time**. This is very useful because the quantization of this system is well understood from several points of view [remember that we can quantize in a Fock space].
- Here it is important to introduce the concept of **"quantum configuration space"**.
  - For a quantum mechanical system with a finite number of d.o.f. this is just the configuration space of the classical system. An example: For a particle in a Coulomb potential (hydrogen atom) the Hilbert space is $L^2(\mathbb{R}^3, \mathrm{d}\mu)$, (square integrable functions $\psi : \mathbb{R}^3 \to \mathbb{C}$ with respect to the Lebesgue measure in $\mathbb{R}^3$). (Remember that the scalar product of $\psi_1, \psi_2 \in L^2(\mathbb{R}^3, \mathrm{d}\mu)$ is $\int_{\mathbb{R}^3} \bar{\psi}_1 \psi_2 \mathrm{d}\mu$). The quantum configuration space is just $\mathbb{R}^3$.
  - What would be the analogous choice for an infinite dimensional quantum system (field theory)?

- Let us consider a real scalar field defined in $\mathbb{R}^3$ (it is not necessary to introduce a concrete dynamics at this point but one can do so.)
- We require that ϕ is sufficiently smooth (for example $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, smooth functions of compact support). **This is the classical configuration space for this system:** $\mathcal{C}_{KG}$.

CAN WE USE IT AS OUR QUANTUM CONFIGURATION SPACE?

NO

- One would expect that the Hilbert space would be $L^2(\mathcal{C}_{KG}, \mathrm{d}\mu)$ but we have no obvious physically plausible choice of measure in this space (the are subtleties too, for example there are no translation invariant measures in infinite dimensional topological vector spaces).
- There is useful procedure to construct an integration theory in infinite dimensional spaces that can be used here and generalized for field theories of connections.

Cylindrical functions $\Psi : \mathcal{C}_{KG} \to \mathbb{R}$

- Let $\mathcal{S}$ be a set of smooth *probe functions* $e : \mathbb{R}^3 \to \mathbb{R}$ of rapid decay and consider the linear functionals $h_e[\phi] : \mathcal{C}_{KG} \to \mathbb{R}; \phi \mapsto \int_{\mathbb{R}^3} e\phi \mathrm{d}\mu$.
- Given a probe e some partial information on ϕ can be obtained by h_e. For instance, if e is "peaked" around a certain point x_0, $h_e(\phi)$ allows us to get the approximate value of $\phi(x_0)$.
- A function $\Psi : \mathcal{C}_{KG} \to \mathbb{R}$ will be called **cylindrical** if there exists a finite number of functions $e_1, \ldots, e_n \in \mathcal{S}$ and a smooth $\psi : \mathbb{R}^n \to \mathbb{R}$ such that for all $\phi \in \mathcal{C}_{KG}$ we have $\Psi(\phi) = \psi(h_{e_1}[\phi], \ldots, h_{e_n}[\phi])$. In such case we say that Ψ is cylindrical w.r.t. $e_1, \ldots, e_n \in \mathcal{S}$.
- They are called cylindrical because do not depend on all the "variables $\phi(x)$" but only on those probed or selected by the specific choice of e's (like $f(x_1, x_2, x_3) = x_1 + x_2^3$).
- Cylindrical functions w.r.t. a fixed set of probes $\alpha = (e_1, \ldots, e_n)$ span a $\mathbb{R}$-vector space that we denote Cyl_α.

The scalar field

- It is easy to turn Cyl_α into a Hilbert space. By taking a measure μ_n in $\mathbb{R}^n$ we define for $\Psi_1, \Psi_2 \in Cyl_\alpha$ the scalar product

$$\langle \Psi_1, \Psi_2 \rangle = \int_{\mathbb{R}^n} \bar{\psi}_1 \psi_2 \, \mathrm{d}\mu_n$$

We consider now the much bigger space of *all* cylindrical functions $\cup_\alpha Cyl_\alpha$ (i.e. cylindrical w.r.t. some set of probes). We want to extend the previous $\langle \cdot, \cdot \rangle$ to this space.

- Some compatibility issues arise (a function can be cylindrical w.r.t several sets of probes) and hence we must check that in these cases the inner product is independent of the set of probes that we use to define it. [notice that $\prod_{i=1}^n \mu_i^{Leb}$ will not work because $(\mathbb{R}, \mu^{Leb})$ is not a finite measure space]
- These compatibility conditions can be met (Gaussian measures).

- A given family of compatible measures $\{\mu_n\}_{n\in\mathbb{N}}$ allows us to define a scalar product for any pair of cylindrical functions.
- There may be many of them!, if we want to select one it is usually helpful to use some symmetry, such as Poincaré if we have the Minkowski metric.
- The Cauchy completion of $(Cyl, \langle , \rangle)$ will be taken as the Hilbert space of quantum states $\mathcal{H}$ for the scalar field.

How do the elements of $\mathcal{H}$ look like?, Is it just $L^2(\mathcal{C}_{KG}, \mathrm{d}\mu)$?

- They are not functions on $\mathcal{C}_{KG}$ but rather functions on the topological dual $\mathcal{S}'$ of the space of probes $\mathcal{S}$ (tempered distributions, notice that $\mathcal{C}_{KG} \subset \mathcal{S}'$). This is now the **quantum configuration space**.
- The Hilbert space is, in fact, given by $\mathcal{H} = L^2(\mathcal{S}', \mathrm{d}\mu)$ where $\mathrm{d}\mu$ is a regular Borel measure on $\mathcal{S}'$ that is an extension of the cylindrical measure used in the construction.
- There is a *duality* between the probes and the functions in the quantum configuration space.

- The classical configuration space $\mathcal{C}_{KG}$ is dense in $\mathcal{S}'$ but $\mu(\mathcal{C}_{KG}) = 0$!
- The representation of configuration and momentum operators in this space is straightforward:
  - Configuration operators act by multiplication.

$$\varphi_f := \int_{\mathbb{R}^3} f \varphi \mathrm{d}\mu^{Leb} \rightarrow (\hat{\varphi}_f.\Psi)[\varphi] = \varphi_f \Psi[\varphi]$$

  - Momentum operators are given by derivations in the ring *Cyl* (These derivations can be seen as vector fields in the quantum configuration space.)
- This quantization of the scalar field is equivalent to the Fock representation mentioned above.

The **main difference** between this example and the standard situation in the quantum mechanics of a system with a finite number of d.o.f. is the necessity to **enlarge the classical configuration space**.

The idea is to follow the steps described above for the scalar field to work with connections

- Let us suppose that the we have a theory of $SU(2)$ connections on a spatial (3-dim) manifold Σ (that is our classical configuration space is that of general relativity in terms of Ashtekar variables).
- We introduce a set of probe functions as in the scalar case to start by reducing our problem to one with a finite number of d.o.f.
  - Gauge invariance suggests that we use gauge invariant probes.
  - A natural choice is to use **holonomies** of the connection along curves in Σ.
  - Using holonomies around a suitably large set of curves (graphs) we can recover all the gauge invariant information contained in the $SU(2)$ connection.
- Once the construction for cylindrical functions of connections is understood generalize it to as for the scalar field. This will force us to find appropriate measures in the completion of *Cyl* and eventually select one by using some sensible criterion.

- A beautiful result. Diff-invariance (plus some plausible conditions on the representation of the algebra of elementary variables) singles out one:

The Ashtekar-Lewandowski measure μ_{AL}.

- This is the content of the LOST&F uniqueness theorems.

- Let us look at the construction of the Hilbert space in some detail.
- **Some preliminary material**
  - $SU(2)$ connections.
  - Holonomies.
  - The Haar measure.
  - Quantum mechanics on SU(2)

$SU(2)$ Connections

- We restrict ourselves to 3-dimensional orientable spatial manifolds Σ so principal $SU(2)$ bundles are trivial. This allows us to represent connections on the bundle by $\mathfrak{su}(2)$-valued 1-form fields ${A_{aA}}^B$ (in the fundamental representation of $SU(2)$).
- It is convenient to write ${A_{aC}}^D = A^i_a {\tau_{iC}}^D$ with the three Lie algebra vectors $\tau_i := \frac{1}{2i}\sigma_i$ ($[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$).

Holonomies

- Connections tell us how to define parallel transport, in the present case given a connection on the $SU(2)$ bundle over Σ and a smooth path $\gamma : [0,1] \to \Sigma$ from p to q in Σ a vector $u(t)$ on the fiber over $\gamma(t)$ is parallel transported along the curve if

$$\frac{d}{dt}u(t) + A(\gamma'(t))u(t) = 0, \quad \left[\frac{d}{dt}u_A(t) + \dot{\gamma}^a(t){A_{aA}}^B(\gamma(t))u_B(t) = 0\right]$$

This can be solved as $[u_0 := u(0)]$

$$u(t) = \sum_{n=0}^{\infty} \left[(-1)^n \int_{t \geq t_1 \geq \cdots \geq t_n \geq 0} A(\gamma'(t_1)) \cdots A(\gamma'(t_n)) \mathrm{d}t_n \cdots \mathrm{d}t_1 \right] u_0$$

$$:= \left\{ \mathcal{P}Exp[-\int_0^t A(\gamma'(s))\mathrm{d}s] \right\} u_0 := h_\gamma[A] u_0$$

The linear map $h_\gamma[A] : V_p \to V_q; u \mapsto h_\gamma[A]u$ is called the **holonomy** along the path γ joining the points p and q.

- It plays a fundamental role in gauge theories.
- Under the action of local gauge transformations on the connection it transforms as $h_\gamma[A'] = g(q) h_\gamma[A] g(p)^{-1}$.
- The holonomy around a *loop* (a curve from p to p) is an endomorphism of the fiber over p, and its trace, known as the **Wilson loop** $W_\gamma[A] := \mathrm{Tr} h_\gamma[A]$ is gauge invariant.

Haar measures.

- Any compact, Hausdorff, topological group has a unique (up to constant factors) left and right invariant measure μ_H.
- Compact Lie groups are finite measure spaces with the Haar measure. [In the following we will normalize them so that $\mu_H(G) = 1$].

Quantum mechanics on $SU(2)$.

Let us consider the quantization of a mechanical system with $SU(2)$ **as its configuration space.** We consider then the Hilbert space $L^2(SU(2), \mathrm{d}\mu_H)$ (with the scalar product defined with the help of the Haar measure).

- **Configuration operators** (smooth functions on $F : SU(2) \to \mathbb{C}$) act by multiplication: $(\hat{f} \cdot \Psi)(g) = f(g)\Psi(g)$.
- **Momentum operators** are associated to smooth vector fields on $SU(2)$. Given a vector field $X \in \mathfrak{X}(SU(2))$ the corresponding momentum operator is defined as the Lie derivative along X [plus a divergence term w.r.t. the invariant volume form in $SU(2)$]:

$$(\hat{P}_X.\Psi)(g) = i[\mathcal{L}_X\Psi + \frac{1}{2}(\mathrm{div}X)\Psi](g).$$

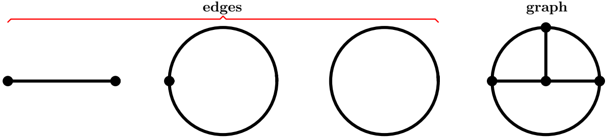
- For a given element in the Lie algebra $v \in \mathfrak{su}(2)$ it is possible to define two natural left and right invariant vector fields L_v and R_v on $SU(2)$.
- If we consider the left and right invariant vector fields naturally defined by each element of the orthonormal basis (w.r.t. the Cartan-Killing metric η_{ij}) given by τ_i, $i = 1, 2, 3$ they define the (commuting) operators $\hat{L}_i$ and $\hat{R}_i$. The divergence terms corresponding to these are zero.
- An interesting operator in $L^2(SU(2), \mathrm{d}\mu_H)$ is the quantum Hamiltonian

$$\hat{J}^2 = \eta_{ij}\hat{L}_i\hat{L}_j = \eta_{ij}\hat{R}_i\hat{R}_j = -\triangle$$

This is describes the dynamics of a free particle on $SU(2)$, i.e. the motion about the center of mass of a solid with $I_1 = I_2 = I_3$ (spherical top).

- There is a useful orthogonal decomposition of the Hilbert space $L^2(SU(2), \mathrm{d}\mu_H)$. For a given vector $v \in \mathfrak{su}(2)$ the set of operators $\{\hat{J}^2, \hat{L}_v, \hat{R}_v\}$ commute. This means that we can find an orthonormal basis of simultaneous eigenstates $D^{(j)}_{m,n}$ of these.
  - This orthonormal basis of $L^2(SU(2), \mathrm{d}\mu_H)$ is given by functions $D^{(j)}_{mm'} : SU(2) \to \mathbb{C} : g \mapsto D^{(j)}_{mm'}(g)$, $j \in \frac{1}{2}\mathbb{N} \cup \{0\}$, and for each j we have $m, m' \in \{-j, -j+1, \ldots, j-1, j\}$.
  - $\langle D^{(j)}_{mm'}, D^{(\ell)}_{nn'} \rangle = \displaystyle\int_{SU(2)} \overline{D}^{(j)}_{mm'}(g) D^{(\ell)}_{nn'}(g) \mathrm{d}\mu_H = \delta_{j\ell}\delta_{mn}\delta_{m'n'}$.
  - These are eigenfunctions with eigenvalue $j(j+1)$ of $-\Delta$ where Δ is the Laplacian on $SU(2)$ defined with the help of η_{ij}.
  - m and m' are the eigenvalues of $\hat{L}_v$ and $\hat{R}_v$.
  - This is a consequence of the Peter-Weyl theorem (an important result of harmonic analysis on groups).

A **graph** is defined as a finite set of **edges** (compact, 1-dimensional, analytic, oriented, embedded submanifolds of Σ) that only intersect in the end points.



Connection dynamics on a graph γ

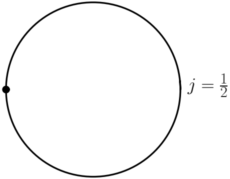
- Consider a **fixed graph** γ on Σ with n_γ edges and v_γ vertices, and restrict (i.e. pull-back) connections and gauge transformations to γ.
- For each edge e_I, $I = 1, \ldots, n_\gamma$ in γ we can get the holonomy of the connection $h_{e_I}[A] \in SU(2)$ so a $SU(2)$ connection on Σ defines a map $A_\gamma : \gamma \to [SU(2)]^{n_\gamma}$.
- Gauge transformations "act" only on the vertices of the graph.

- Given a $SU(2)$ connection on Σ we can get a finite number of group elements associated to each edge and we are left with *residual gauge transformations* at the vertices of the graph. We will use this construction to define appropriate cylindrical functions for connection theories.
- Let us fix a graph γ with n_γ edges and v_γ vertices.
- **Definition of** Cyl_γ
  - Let us consider as our configuration space the space $\mathcal{A}$ of smooth $SU(2)$ connections on Σ.
  - We will say that a function $\Psi : \mathcal{A} \to \mathbb{C}$ is cylindrical if there is a graph γ with n_γ edges and a function $\psi : [SU(2)]^{n_\gamma} \to \mathbb{C}$ such that $\Psi(A) = \psi(h_{e_1}[A], \ldots, h_{e_{n_\gamma}}[A])$.
- Cylindrical functions depend on the connection A only trough its holonomies along the edges of γ. The holonomies along the edges play the role of "gauge covariant probes" to obtain (partial) information about the connections on Σ.

Connection field theories

Examples of cylindrical functions associated to graphs

1. The Wilson loop:
   - Consider a loop γ

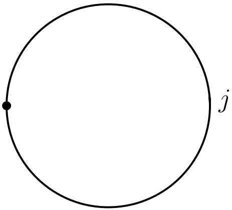


   - Take the **trace of the holonomy** around γ, $W_\gamma[A] := Tr[h_\gamma[A]]$ in the fundamental representation ($j = 1/2$). (The result is independent of the point that you choose, so it really corresponds to an edge without a marked point).
   - If $h_\gamma[A] = \begin{bmatrix} a+id & c+ib \\ -c+ib & a-id \end{bmatrix} \rightarrow W_\gamma[A] = 2a$

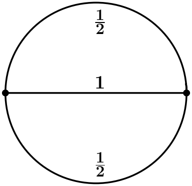
   $(a^2 + b^2 + c^2 + d^2 = 1)$

2. There is no need to restrict oneself to the fundamental representation of $SU(2)$. If one takes the representation $\mathfrak{D}^{(j)}$, $j \in \frac{1}{2}\mathbb{N} \cup \{0\}$ given by unitary matrices in $\mathcal{M}_{2j+1}(\mathbb{C})$ one can compute $W^j_\gamma[A] := Tr[\mathfrak{D}^{(j)}(h_\gamma(A))]$.

   - In this case we label the loop γ with j



3. We have now $D^{(1/2)}_{AB}(h_{e_1}(A))$, $D^{(1)}_{ij}(h_{e_2}(A))$, $D^{(1/2)}_{CD}(h_{e_3}(A))$ where the indices $A; B; C; D \in \{-1/2, 1/2\}$, $i, j \in \{-1, 0, 1\}$.



- To get a complex number we can just choose some fixed components for each of the matrices (in which case the function would not be invariant under $SU(2)$ transformations on the vertices).

- If we want to get a gauge invariant object wee must contract the indices with a $SU(2)$ invariant object in $\mathfrak{D}^{(1/2)} \otimes \mathfrak{D}^{(1)} \otimes \mathfrak{D}^{(1/2)}$. [this is called an **intertwinner** and can be constructed from symmetrized products of the antisymmetric objects ϵ_{AB} and ϵ^{AB}].
- In this case the (essentially) unique choice is $\sigma^i_{AC}\sigma^j_{BD}$ (for vertices of degree 3 these objects can be written in terms of Clebsh-Gordan coefficients or Wigner $3j$ symbols. For higher degrees there are many more choices, one has to consider nj symbols).

Comments

- A cylindrical function w.r.t. a graph γ is also cylindrical w.r.t. any graph γ' that contains γ. We will have to take this into account when we define orthonormal bases on the final Hilbert space.
- We want to turn the vector space Cyl_γ into a Hilbert space.
- The role played by Gaussian measures for the scalar field is played now by the Haar measure on $SU(2)$. Notice the importance of the fact that $SU(2)$ is compact!
- For two functions Ψ_1, Ψ_2 in Cyl_γ we define the **scalar product**

$$\langle \Psi_1, \Psi_2 \rangle = \int_{[SU(2)]^{n_\gamma}} \prod_{j=1}^{n_\gamma} \mathrm{d}\mu_H(g_j) \bar{\psi}_1(g_1, \ldots, g_{n_\gamma}) \psi_2(g_1, \ldots, g_{n_\gamma})$$

- $\prod_{j=1}^{n} \mathrm{d}\mu_H(g_j)$ is the Haar measure on $[SU(2)]^{n_\gamma}$
- The Hilbert space for a given graph γ will be called $\mathcal{H}_\gamma$.

Useful operators in $\mathcal{H}_\gamma$

- Given a graph we have built a Hilbert space as a tensor product of the $L^2(SU(2), \mathrm{d}\mu_H)$ associated to each edge.
- We can build operators in $\mathcal{H}_\gamma$ from those defined in each $L^2(SU(2), \mathrm{d}\mu)$. These are useful to label orthogonal subspaces in a direct sum decomposition of $\mathcal{H}_\gamma$, label the elements of useful orthonormal bases, and also to build the geometric operators associated to areas and volumes.
- **Operators associated to edges:**
  - Choose an (oriented) edge e_I, one of the vertices v of e_I, and a basis τ_i in $\mathfrak{su}(2)$. We define $\hat{J}_i^{(v,e_I)}$ as an operator acting on $L^2(SU(2), \mathrm{d}\mu)_{e_I}$. It is $\hat{L}_i$ if e_I starts at v and $\hat{R}_i$ if e_I ends at v.
  - $\hat{J}_e^2 := \eta^{ij}\hat{J}_i^{(v,e)}\hat{J}_j^{(v,e)}$ with eigenvalues $j_e(j_e+1)$, $j_e \in \frac{1}{2}\mathbb{N}\cup\{0\}$. Those corresponding to different edges obviously commute.
  - We can use these to write $\mathcal{H}_\gamma = \oplus_{j_{e_I}}\mathcal{H}_{\gamma, j_{e_I}}$ ($\dim \mathcal{H}_{\gamma, j_{e_I}} > 1$ generically).

- **Operators associated to vertices**
  - Choose a vertex v and consider all the edges $\tilde{e}$ leaving or arriving at it.
  - Define $\hat{J}_i^v = \sum_{\tilde{e} @ v} \hat{J}_i^{(v,\tilde{e})}$ and $\hat{J}_v^2 = \eta_{ij} \hat{J}_i^v \hat{J}_j^v$. These operators have eigenvalues $j_v(j_v+1)$ and commute with the $\hat{J}_e^2$.
  - We can use these them to further split $\mathcal{H}_\gamma$ as $\mathcal{H}_\gamma = \oplus_{j_{e_I}, j_{v_\ell}} \mathcal{H}_{\gamma, j_{e_I}, j_{v_\ell}}$.
  - We can build other operators at the vertices by considering only a subset of the edges arriving at any one of them. These can be used to further decompose the subspaces $\mathcal{H}_{\gamma, j_{e_I}, j_{v_\ell}}$.

The definition of Cyl for SU(2) connection theories

- As before we consider now the space of all cylindrical functions w.r.t. any graph $Cyl = \cup_\gamma Cyl_\gamma$. A very large space.
- In order to define the **scalar product** for any pair of cylindrical functions (associated to possibly different graphs γ_1 and γ_2) we:
  - Introduce a third graph γ_3 such that $\gamma_1 \subset \gamma_3$ and $\gamma_2 \subset \gamma_3$.
  - Both cylindrical functions are cylindrical w.r.t. γ_3.
  - Use the previous definition for γ_3.
  - The procedure gives a unique result independent of the choice of γ_3 owing to the left and right invariance of the Haar measure and the fact that we choose it normalized (the compatibility conditions relevant in this case can be met).
  - The scalar product of cylindrical functions associated to different graphs is automatically zero.
- Our kinematical Hilbert space will be the **Cauchy completion** $\overline{Cyl}$ of Cyl with this scalar product.

Characterization of the elements of $\overline{Cyl}$:

Can we think of it as some $L^2(\bar{\mathcal{A}}, \mathrm{d}\mu)$?

YES

- What is the quantum configuration space $\bar{\mathcal{A}}$? (there are different characterizations:) we give one here:
- Its elements are *quantum connections* $\bar{A}$ that assign to any $e \in \Sigma$ an element in $SU(2)$ such that:
  - $\bar{A}(e_2 \circ e_1) = \bar{A}(e_2)\bar{A}(e_1)$, $\bar{A}(e^{-1}) = (\bar{A})^{-1}(e)$
  - No other conditions (in particular no continuity requirements!)
- Given any quantum connection and any graph there is a smooth connection A such that $\bar{A}(e) = A(e)$ for all the edges.
- This Hilbert space is non-separable ("very big"). **Can we really handle it?** Roughly speaking the big size problem is somehow tamed by the fact that we have diff-invariance.

- This space has another beautiful description: The Abelian algebra of cylindrical functions can be extended to an abelian C^*-algebra with unit $\overline{Cyl}$. This can be represented as the space of continuous functions over a compact, Haussdorf space called (the spectrum of the algebra $sp(\overline{Cyl})$). We have $\bar{\mathcal{A}} = sp(\overline{Cyl})$

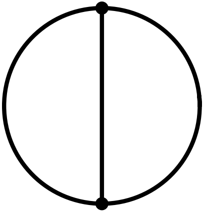
What about the measure?

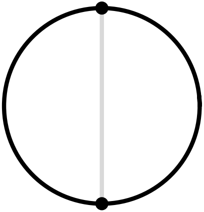
- There are compatibility issues of the type described for the scalar field but complicated by the fact that our setting is intrinsically non-linear now. Technically one has to use projective techniques.
- The family of induced Haar measures that we have introduced for each graph defines a regular Borel measure on $\bar{\mathcal{A}}$ which is **invariant** under the natural action of **diffomeorfisms** on Σ. This is known as the **Ashtekar-Lewandowski measure**. A remarkable result and somehow unexpected due to the results on the non-existence of translation invariant measures in topological vector spaces.

- One can in principle construct several families of diff-invariant measures associated to knot invariants.
- μ_{AL} is unique under natural assumptions (LOST theorem) (after the introduction of the holonomy-flux algebra).
- In the Gel'fand topology in $\bar{\mathcal{A}}$ the space of smooth connections $\mathcal{A}$ is densely embedded in $\bar{\mathcal{A}}$ [Marolf&Mourão] but has zero measure w.r.t. μ_{AL}!
- The Hilbert space $L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_{AL})$ carries a natural representation the group of $SU(2)$ gauge transformations and diffeomorphisms [with some technical qualifications]. The scalar product is **invariant** under these and the representation is **unitary**.

An orthonormal basis on $\mathcal{H}$

- It is not possible to directly write $\mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma$ because the Hilbert space $\mathcal{H}_{\gamma_2}$ is a non-trivial subspace of $\mathcal{H}_{\gamma_1}$ if $\gamma_2 \subset \gamma_1$.
- This can be easily solved by taking $\mathcal{H}'_\gamma$, the subspace of $\mathcal{H}_\gamma$ orthogonal to the subspace $\mathcal{H}'_{\tilde{\gamma}}$ associated to every $\tilde{\gamma}$ strictly contained in γ.
- There is a simple characterization of this space in terms of the eigenvalues of the operators $\hat{J}_e^2$ associated to the edges of γ and the $\hat{J}_v^2$ associated to any spurious vertices that may be present (vertices that do nothing but "split an edge at a point where it is analytic").





A useful orthonormal basis

- For a fixed graph γ the Peter-Weyl theorem tells us that the product of the functions $D^{(j_1)}_{m_1 n_1}(g_1)\cdots D^{(j_{n_\gamma})}_{m_{n_\gamma} n_{n_\gamma}}(g_{n_\gamma})$, $j_1,\ldots,j_{n_\gamma} \in \frac{1}{2}\mathbb{N}\cup\{0\}$, m_i, $n_i \in j_i+\mathbb{Z}$ with $-j_i \leq m_i, n_i \leq j_i$ are the elements of an orthonormal basis of $\mathcal{H}_\gamma$.
- The orthonormal basis in the full Hilbert space $\mathcal{H}$ would then be:

$$\bigcup_\gamma \left\{ D^{(j_1)}_{m_1 n_1}\otimes\cdots\otimes D^{(j_{n_\gamma})}_{m_{n_\gamma} n_{n_\gamma}} : j_i \in \frac{1}{2}\mathbb{N},\ m_i,\ n_i \in j_i+\mathbb{Z},\ -j_i \leq m_i, n_i \leq j_i \right\}$$

(notice that the $j_i \neq 0$)

- The theory of angular momentum in angular mechanics can be used to look for other bases associated to other commuting sets of angular momentum operators of the type described before.