# Optimal control problems for affine connection control systems: characterization of extremals

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# Outline

1. Optimal Control Problem for Affine Connection Control Systems

2. Presymplectic Constraint Algorithm for ACCS

3. Application: Time-Optimal Control Problem, $F = 1$

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## Affine Connection Control System (ACCS)

Let $Q$ be a smooth manifold, $\dim Q = n$.
Let $\nabla$ be an affine connection on $Q$.
Consider the control system

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = u^k(t)Y_k(\gamma(t)),$$

where

- $\gamma\colon I \subset \mathbb{R} \to Q$ is a curve,
- $u\colon I \to U \subset \mathbb{R}^m$ are locally integrable *controls*,
- $U$ is an open set,
- $Y_k$ are *input vector fields* on $Q$.

An *Affine Connection Control System* is $\Sigma = (Q, \nabla, \mathscr{Y}, U)$,

where $\mathscr{Y} = \{Y_1, \dots, Y_m\}$.

The above second-order equation is rewritten on $TQ$,

$$\dot{\Upsilon}(t) = Z(\Upsilon(t)) + u^k(t) Y_k^V(\Upsilon(t)), \quad X = Z + u^k Y_k^V,$$

where

- $\Upsilon \colon I \to TQ$ is a curve such that $\Upsilon = \dot{\gamma}$,
- $Z$ is the *geodesic spray* associated to $\nabla$, a vector field on $TQ$. In natural coordinates $(x, v)$ for $TQ$,

$$Z = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jl}(x) v^j v^l \frac{\partial}{\partial v^i}, \quad \Gamma^i_{jl} \text{ Christoffel symbols for } \nabla\ .$$

- $Y_k^V$ denotes the vertical lift of the vector field $Y_k$.

# Free-time Optimal Control Problem for ACCS (OCP)

Let $F\colon TQ \times U \to \mathbb{R}$ be a *cost function*.

Given $\Sigma = (Q, \nabla, \mathscr{Y}, U)$, $F$.

*Find* $I = [a, b] \subset \mathbb{R}$ *and* $(\gamma, u)\colon I \to Q \times U$

such that there exists $\Upsilon\colon I \to TQ$ along $\gamma$ satisfying

(1) $\Upsilon(a) = v_{x_a}$, $\Upsilon(b) = v_{x_b}$, given $v_{x_a} \in T_{x_a}Q$, $v_{x_b} \in T_{x_b}Q$,

(2) $\dot{\Upsilon}(t) = (Z + u^k Y_k^V)(\Upsilon(t)) \quad (\Rightarrow \Upsilon = \dot{\gamma})$,

(3) $\mathcal{S}[\Upsilon, u] = \int_I F(\Upsilon(t), u(t))dt$ is minimum over all curves on $TQ \times U$ satisfying (1) and (2).

## Presymplectic formalism in OCP

Let $M$ be a smooth manifold and $\pi_1 \colon T^*M \times U \to T^*M$.

Let $(T^*M \times U, \Omega)$ be the *presymplectic manifold*, where

$\Omega$ is the $\pi_1$-pullback of the natural 2-form in $T^*M$.

In natural coordinates $(x, p, u)$ for $T^*M \times U$,

$$\Omega = \mathrm{d}p_i \wedge \mathrm{d}x^i, \quad \ker \Omega = \left\{ \frac{\partial}{\partial u^k} \right\}_{k=1,\dots,m}.$$

# Presymplectic formalism in OCP

Let $X$ be a vector field along $\pi\colon M \times U \to M$,
the cost function $F\colon M \times U \to \mathbb{R}$ and $p_0 \in \{-1, 0\}$,

we define the *Hamiltonian* $H\colon T^*M \times U \to \mathbb{R}$,

$$H(p, u) = (H_X + p_0 F)(p, u) = \langle p, X(x, u)\rangle + p_0 F(x, u), \quad p \in T^*_x M.$$

Then $(T^*M \times U, \Omega, H)$ is a *presymplectic Hamiltonian system*

and $\quad i_{X_H}\Omega = \mathrm{d}H \quad$ is the *presymplectic equation*.

# Now

- $M = TQ$,
- $X = Z + u^k Y_k^V \in \mathfrak{X}(TQ)$,
- $H\colon T^*(TQ) \times U \to \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic

  Hamiltonian system in OCP for ACCS.

$$
\begin{tikzcd}
 & & T^*(TQ) \arrow[d, "\pi_{TQ}"] \\
 & & TQ \arrow[d, "\tau_Q"] \\
I \arrow[urruu, "\Lambda"] \arrow[urr, "\dot{\gamma} = \Upsilon"'] \arrow[rr, "\gamma"'] & & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*(TQ) \times U \arrow[d] \\
TQ \times U \arrow[d] \\
Q \times U
\end{tikzcd}
$$

# Weak Pontryagin's Maximum Principle (PMP)

## Theorem

*Let $(\Upsilon, u)\colon [a,b] \to TQ \times U$ be a solution of OCP with initial conditions $v_{x_a}$, $v_{x_b}$. Then there exist $\Lambda\colon [a,b] \to T^*(TQ)$ along $\Upsilon$, and a constant $p_0 \in \{-1, 0\}$ such that:*

1. *$(\Lambda, u)$ is an integral curve of the Hamiltonian vector field $X_H$,* $\quad i_{X_H}\Omega = \mathrm{d}H$;
2. $\Upsilon = \pi_{TQ} \circ \Lambda$, *where* $\pi_{TQ}\colon T^*(TQ) \to TQ$;
3. $\Upsilon$ *satisfies the initial conditions in* $TQ$;
4. (A) $\max_{\widetilde{u} \in U} H(\Lambda(t), \widetilde{u}) = 0$ *for* $t \in [a,b]$;
   (B) $(p_0, \Lambda(t)) \neq 0$ *for each* $t \in [a,b]$.

# Different kinds of extremals

## Definition

A curve $(\Upsilon, u)\colon [a,b] \to TQ \times U$ for OCP is

1. an extremal if there exist $\Lambda\colon [a,b] \to T^*(TQ)$ and a constant $p_0 \in \{-1, 0\}$ such that $\Upsilon = \pi_{TQ} \circ \Lambda$ and $(\Lambda, u)$ satisfies the necessary conditions of PMP;
2. a normal extremal if it is an extremal and $p_0 = -1$;
3. an abnormal extremal if it is an extremal and $p_0 = 0$;
4. a strictly abnormal extremal if it is not a normal extremal, but it is an abnormal extremal.

The curve $(\Lambda, u)\colon [a,b] \to T^*(TQ) \times U$ along $\Upsilon$ is called biextremal for *OCP*.

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## Presymplectic Constraint Algorithm (Gotay-Nester)

Given $(M, \Omega, H)$ and $i_X\Omega = \mathrm{d}H$, find $(N, X)$ such that

(A) $N$ is a submanifold of $M$,

(B) $X$ is a vector field tangent to $N$,

(C) $N$ is maximal among all the submanifolds satisfying $A$, $B$.

Primary constraint submanifold

$$\begin{aligned} N_0 &= \{x \in M \,|\, \exists\, v_x \in T_xM\,,\ i_{v_x}\Omega = \mathrm{d}_xH\} \\ &= \{x \in M \,|\, Z(H)(x) = 0\,,\ \forall Z \in \ker\Omega\} \\ X^{N_0} &= X^0 + \ker\,\Omega,\ X^0 \text{ is a solution of } i_X\Omega = \mathrm{d}H \end{aligned}$$

Stabilization: $N_1 = \{x \in N_0 \,|\, \exists\, X \in X^{N_0}\,,\ X(x) \in T_xN_0\}$.

$$(N_i, X^{N_i}), \quad N_{i+1} = \{x \in N_i \,|\, \exists\, X \in X^{N_i}\,,\ X(x) \in T_xN_i\}.$$

If $\exists\, i \in \mathbb{N}$ such that $N_i = N_{i-1}$,

$N_f = N_{i-1}$ is the final constraint submanifold.

# Now in OCP for ACCS

- $M = T^*(TQ) \times U$,
- $H\colon T^*(TQ) \times U \to \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic Hamiltonian system in OCP for ACCS,
- $i_{X_H}\Omega = dH$ and locally

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} + C^k \frac{\partial}{\partial u^k}.$$

## Constraint Algorithm in OCP for ACCS (free-time)

Primary submanifold

$$N_0 = \left\{ (\Lambda, u) \in T^*(TQ) \times U \;\middle|\; \begin{array}{l} \overbrace{H_{Y_k^V} + p_0 \dfrac{\partial F}{\partial u^k}}^{\frac{\partial H}{\partial u^k}=} = 0, \;\; k = 1, \ldots, m \\ H = 0. \end{array} \right\}$$

First stabilization step:

$N_1 = \{(\Lambda, u) \in N_0 \,|\, X_H(\Lambda, u) \in T_{(\Lambda,u)} N_0\}.$

Tangency conditions:

$$X_H\left(H_{Y_k^V} + p_0 \frac{\partial F}{\partial u^k}\right) = 0,$$

$$X_H(H) = 0 \quad \text{Trivially.}$$

| Normality | Abnormality |
|---|---|
| $p_0 = -1$ | $p_0 = 0$ |
| $\{H_{Y_k^V} = \frac{\partial F}{\partial u^k}, H = 0\}(= N_0^{[-1]})$ | $\{H_{Y_k^V} = 0, H = 0\}(= N_0^{[0]})$ |
| $N_1^{[-1]}$ | $N_0^{[0]} \cap \{H_{[Z, Y_k^V]} = 0\}(= N_1^{[0]})$ |
| $\vdots$ | $\vdots$ |
| $(N_f^{[-1]}, X_f^{[-1]})$ | $(N_f^{[0]}, X_f^{[0]})$<br>Delete zero covector |

## Strict abnormality

Let $\rho\colon T^*(TQ)\times U \to TQ\times U$ and $\mathbf{P} = \rho(N_f^{[0]}) \cap \rho(N_f^{[-1]})$.

| | | |
|---|---|---|
| $\mathbf{P}=\emptyset$ | $\rho(N_f^{[0]}) \neq \emptyset$ | all the abnormal extremals are strict. |
| | $\rho(N_f^{[-1]}) \neq \emptyset$ | all the normal extremals are strict normal. |
| $\mathbf{P}\neq\emptyset$ | $\mathbf{P} = \rho(N_f^{[0]})$ | no strict abnormal extremals. |
| | $\mathbf{P} \neq \rho(N_f^{[0]})$ | local strict abnormal extremals. |
| | $\mathbf{P} = \rho(N_f^{[0]}) = \rho(N_f^{[-1]})$ | all the abnormal extremals are also normal and viceversa. |

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## Constraint Algorithm for Time-Optimal Problem, $F = 1$

Pontryagin's Hamiltonian $\quad H = H_Z + u^k H_{Y_k^V} + p_0.$

On the submanifold $H = 0$, we obtain $N_f^{[-1]}$ and $N_f^{[0]}$.

Put condition $H = 0$ aside and apply the algorithm:

$$N_0 = N_0^{[0]} = N_0^{[-1]} = \{(\Lambda, u) \in T^*(TQ) \times U \,|\, H_{Y_k^V} = 0\},$$
$$N_1 = \{(\Lambda, u) \in N_0 \,|\, H_{[Z, Y_k^V]} = 0\},$$

for $k = 1, \ldots, m$, and so on until $N_f$, if it exists.

The actual final constraint submanifolds are

$$\begin{aligned} N_f^{[0]} &= N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \,|\, H_Z + u^k H_{Y_k^V} = 0\}, \\ N_f^{[-1]} &= N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \,|\, H_Z + u^k H_{Y_k^V} = 1\}. \end{aligned}$$

## Results for Time-Optimal Control Problem, $F = 1$

### Proposition

*Let $\Sigma$ be an ACCS. Given a time-optimal control problem:*

1. *If $N_f^{[0]}$ only has zero covectors, there are no abnormal extremals.*
2. *If $N_f^{[0]}$ has nonzero covectors and $N_f \subset \{(\Lambda, u) \in T^*(TQ) \times U \,|\, (H_Z + u^j H_{Y_j^V}) = 0\}$, then every abnormal extremal is strict and there are no normal extremals.*

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