Optimal control problems for affine connection control systems: characterization of extremals

María Barbero-Liñán, Miguel C. Muñoz-Lecanda

Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya

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Outline

1. Optimal Control Problem for Affine Connection Control Systems
2. Presymplectic Constraint Algorithm for ACCS
3. Application: Time-Optimal Control Problem, $F = 1$

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Affine Connection Control System (ACCS)

Let Q be a smooth manifold, $\dim Q = n$.
Let ∇ be an affine connection on Q.
Consider the control system

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = u^k(t) Y_k(\gamma(t)),$$

where

- $\gamma \colon I \subset \mathbb{R} \to Q$ is a curve,
- $u \colon I \to U \subset \mathbb{R}^m$ are locally integrable *controls*,
- U is an open set,
- Y_k are *input vector fields* on Q.

An *Affine Connection Control System* is $\Sigma = (Q, \nabla, \mathscr{Y}, U)$,

where $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$.

The above second-order equation is rewritten on TQ,

$$\dot{\Upsilon}(t) = Z(\Upsilon(t)) + u^k(t) Y_k^V(\Upsilon(t)), \quad X = Z + u^k Y_k^V,$$

where

- $\Upsilon \colon I \to TQ$ is a curve such that $\Upsilon = \dot{\gamma}$,
- Z is the *geodesic spray* associated to ∇, a vector field on TQ. In natural coordinates (x, v) for TQ,

$$Z = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jl}(x) v^j v^l \frac{\partial}{\partial v^i}, \quad \Gamma^i_{jl} \text{ Christoffel symbols for } \nabla .$$

- Y_k^V denotes the vertical lift of the vector field Y_k.

Free-time Optimal Control Problem for ACCS (OCP)

Let $F\colon TQ \times U \to \mathbb{R}$ be a *cost function*.

Given $\Sigma = (Q, \nabla, \mathscr{Y}, U)$, F.

Find $I = [a, b] \subset \mathbb{R}$ *and* $(\gamma, u)\colon I \to Q \times U$

such that there exists $\Upsilon\colon I \to TQ$ along γ satisfying

(1) $\Upsilon(a) = v_{x_a}$, $\Upsilon(b) = v_{x_b}$, given $v_{x_a} \in T_{x_a}Q$, $v_{x_b} \in T_{x_b}Q$,

(2) $\dot{\Upsilon}(t) = (Z + u^k Y_k^V)(\Upsilon(t)) \quad (\Rightarrow \Upsilon = \dot{\gamma})$,

(3) $\mathcal{S}[\Upsilon, u] = \int_I F(\Upsilon(t), u(t))dt$ is minimum over all curves on $TQ \times U$ satisfying (1) and (2).

Presymplectic formalism in OCP

Let M be a smooth manifold and $\pi_1 \colon T^*M \times U \to T^*M$.

Let $(T^*M \times U, \Omega)$ be the *presymplectic manifold*, where

Ω is the π_1-pullback of the natural 2-form in T^*M.

In natural coordinates (x, p, u) for $T^*M \times U$,

$$\Omega = \mathrm{d}p_i \wedge \mathrm{d}x^i, \quad \ker \Omega = \left\{ \frac{\partial}{\partial u^k} \right\}_{k=1,\dots,m}.$$

Presymplectic formalism in OCP

Let X be a vector field along $\pi\colon M \times U \to M$,
the cost function $F\colon M \times U \to \mathbb{R}$ and $p_0 \in \{-1, 0\}$,

we define the *Hamiltonian* $H\colon T^*M \times U \to \mathbb{R}$,

$$H(p,u) = (H_X + p_0 F)(p,u) = \langle p, X(x,u)\rangle + p_0 F(x,u), \quad p \in T_x^*M.$$

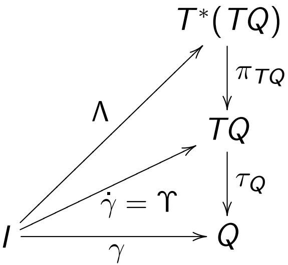
Then $(T^*M \times U, \Omega, H)$ is a *presymplectic Hamiltonian system*

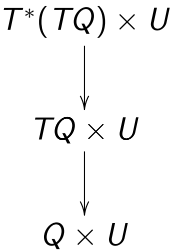
and $\quad i_{X_H}\Omega = \mathrm{d}H \quad$ is the *presymplectic equation*.

Now

- $M = TQ$,
- $X = Z + u^k Y_k^V \in \mathfrak{X}(TQ)$,
- $H\colon T^*(TQ) \times U \to \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic

  Hamiltonian system in OCP for ACCS.





Weak Pontryagin's Maximum Principle (PMP)

Theorem

Let $(\Upsilon, u)\colon [a,b] \to TQ \times U$ *be a solution of OCP with initial conditions* v_{x_a}, v_{x_b}. *Then there exist* $\Lambda\colon [a,b] \to T^*(TQ)$ *along* Υ, *and a constant* $p_0 \in \{-1, 0\}$ *such that:*

1. (Λ, u) *is an integral curve of the Hamiltonian vector field* X_H, $\quad i_{X_H}\Omega = \mathrm{d}H$;
2. $\Upsilon = \pi_{TQ} \circ \Lambda$, *where* $\pi_{TQ}\colon T^*(TQ) \to TQ$;
3. Υ *satisfies the initial conditions in* TQ;
4. (A) $\max_{\widetilde{u} \in U} H(\Lambda(t), \widetilde{u}) = 0$ *for* $t \in [a,b]$;
   (B) $(p_0, \Lambda(t)) \neq 0$ *for each* $t \in [a,b]$.

Different kinds of extremals

Definition

A curve $(\Upsilon, u)\colon [a, b] \to TQ \times U$ for OCP is

1. an extremal if there exist $\Lambda\colon [a, b] \to T^*(TQ)$ and a constant $p_0 \in \{-1, 0\}$ such that $\Upsilon = \pi_{TQ} \circ \Lambda$ and (Λ, u) satisfies the necessary conditions of PMP;
2. a normal extremal if it is an extremal and $p_0 = -1$;
3. an abnormal extremal if it is an extremal and $p_0 = 0$;
4. a strictly abnormal extremal if it is not a normal extremal, but it is an abnormal extremal.

The curve $(\Lambda, u)\colon [a, b] \to T^*(TQ) \times U$ along Υ is called biextremal for *OCP*.

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Presymplectic Constraint Algorithm (Gotay-Nester)

Given (M, Ω, H) and $i_X\Omega = \mathrm{d}H$, find (N, X) such that

(A) N is a submanifold of M,

(B) X is a vector field tangent to N,

(C) N is maximal among all the submanifolds satisfying A, B.

Primary constraint submanifold

$$\begin{aligned} N_0 &= \{x \in M \,|\, \exists\, v_x \in T_xM\,,\ i_{v_x}\Omega = \mathrm{d}_xH\} \\ &= \{x \in M \,|\, Z(H)(x) = 0\,,\ \forall Z \in \ker \Omega\} \\ X^{N_0} &= X^0 + \ker\,\Omega,\ X^0 \text{ is a solution of } i_X\Omega = \mathrm{d}H \end{aligned}$$

Stabilization: $N_1 = \{x \in N_0 \,|\, \exists\, X \in X^{N_0}\,,\ X(x) \in T_xN_0\}$.

$$(N_i, X^{N_i}), \quad N_{i+1} = \{x \in N_i \,|\, \exists\, X \in X^{N_i}\,,\ X(x) \in T_xN_i\}.$$

If $\exists\, i \in \mathbb{N}$ such that $N_i = N_{i-1}$,

$N_f = N_{i-1}$ is the final constraint submanifold.

Now in OCP for ACCS

- $M = T^*(TQ) \times U$,
- $H\colon T^*(TQ) \times U \to \mathbb{R}$, $H = H_Z + u^k H_{Y_k^V} + p_0 F$,
- $(T^*(TQ) \times U, \Omega, H)$ is the presymplectic Hamiltonian system in OCP for ACCS,
- $i_{X_H}\Omega = dH$ and locally

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} + C^k \frac{\partial}{\partial u^k}.$$

Constraint Algorithm in OCP for ACCS (free-time)

Primary submanifold

$$N_0 = \left\{ (\Lambda, u) \in T^*(TQ) \times U \;\middle|\; \begin{array}{l} \overbrace{H_{Y_k^V} + p_0 \dfrac{\partial F}{\partial u^k}}^{\frac{\partial H}{\partial u^k} =} = 0, \; k = 1, \ldots, m \\ H = 0. \end{array} \right\}$$

First stabilization step:

$N_1 = \{(\Lambda, u) \in N_0 \,|\, X_H(\Lambda, u) \in T_{(\Lambda,u)} N_0\}$.

Tangency conditions:

$$X_H(H_{Y_k^V} + p_0 \frac{\partial F}{\partial u^k}) = 0,$$

$$X_H(H) = 0 \quad \text{Trivially.}$$

Normality	Abnormality
$p_0 = -1$	$p_0 = 0$
$\{H_{Y_k^V} = \frac{\partial F}{\partial u^k}, H = 0\}(= N_0^{[-1]})$	$\{H_{Y_k^V} = 0, H = 0\}(= N_0^{[0]})$
$N_1^{[-1]}$	$N_0^{[0]} \cap \{H_{[Z, Y_k^V]} = 0\}(= N_1^{[0]})$
$\vdots$	$\vdots$
$(N_f^{[-1]}, X_f^{[-1]})$	$(N_f^{[0]}, X_f^{[0]})$ Delete zero covector

Strict abnormality

Let $\rho\colon T^*(TQ) \times U \to TQ \times U$ and $\mathbf{P} = \rho(N_f^{[0]}) \cap \rho(N_f^{[-1]})$.

$\mathbf{P} = \emptyset$	$\rho(N_f^{[0]}) \neq \emptyset$	all the abnormal extremals are strict.
	$\rho(N_f^{[-1]}) \neq \emptyset$	all the normal extremals are strict normal.
$\mathbf{P} \neq \emptyset$	$\mathbf{P} = \rho(N_f^{[0]})$	no strict abnormal extremals.
	$\mathbf{P} \neq \rho(N_f^{[0]})$	local strict abnormal extremals.
	$\mathbf{P} = \rho(N_f^{[0]}) = \rho(N_f^{[-1]})$	all the abnormal extremals are also normal and viceversa.

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Constraint Algorithm for Time-Optimal Problem, $F=1$

Pontryagin's Hamiltonian $\quad H = H_Z + u^k H_{Y_k^V} + p_0.$

On the submanifold $H=0$, we obtain $N_f^{[-1]}$ and $N_f^{[0]}$.

Put condition $H=0$ aside and apply the algorithm:

$$N_0 = N_0^{[0]} = N_0^{[-1]} = \{(\Lambda, u) \in T^*(TQ) \times U \mid H_{Y_k^V} = 0\},$$
$$N_1 = \{(\Lambda, u) \in N_0 \mid H_{[Z, Y_k^V]} = 0\},$$

for $k = 1, \ldots, m$, and so on until N_f, if it exists.

The actual final constraint submanifolds are

$$\begin{aligned} N_f^{[0]} &= N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \mid H_Z + u^k H_{Y_k^V} = 0\}, \\ N_f^{[-1]} &= N_f \cap \{(\Lambda, u) \in T^*(TQ) \times U \mid H_Z + u^k H_{Y_k^V} = 1\}. \end{aligned}$$

Results for Time-Optimal Control Problem, $F = 1$

Proposition

Let Σ be an ACCS. Given a time-optimal control problem:

1. *If $N_f^{[0]}$ only has zero covectors, there are no abnormal extremals.*
2. *If $N_f^{[0]}$ has nonzero covectors and $N_f \subset \{(\Lambda, u) \in T^*(TQ) \times U \,|\, (H_Z + u^j H_{Y_j^V}) = 0\}$, then every abnormal extremal is strict and there are no normal extremals.*

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