Modular class and integrability in Poisson and related geometries

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XVI IFWGP 2007

Outline

It was used by...

- J. L. Koszul (1985)
- J. P. Dufour and A. Haraki (1991)
- Z. -L. Liu and P. Xu (1992)
- J. Grabowski, G. Marmo and A. M. Perelomov (1993)

Classify quadratic Poisson structures

What is the Curl Operator?

On a Poisson manifold (M, π) choose a volume form μ:

$$\phi : \mathfrak{X}^k(M) \xrightarrow{\simeq} \Omega^{n-k}(M), \qquad P \longmapsto i_P\mu$$

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- The **curl operator** is defined as

$$\begin{array}{cccccccc} D_\mu : & \mathfrak{X}^k(M) & \to & \Omega^{n-k}(M) & \to & \Omega^{n-k+1}(M) & \to & \mathfrak{X}^{k-1}(M) \\ & P & \stackrel{\phi}{\to} & i_P\mu & \stackrel{\mathrm{d}}{\to} & \mathrm{d}i_P\mu & \stackrel{\phi^{-1}}{\to} & \underbrace{D_\mu P}_{\textbf{curl of P}} \end{array}$$

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- The curl of a vector field is its divergence with respect to μ:

$$D_\mu X = \mathrm{div}_\mu(X) = \frac{\mathcal{L}_X\mu}{\mu}, \quad X \in \mathfrak{X}^1(M).$$

Definition

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Remark

This modular vector field depends on the choice of μ.

Weinstein geometric approach

Question

Given (M, π) is there a volume form μ invariant under the flow of all hamiltonian vector fields, i.e, such that:

$$\mathcal{L}_{X_f}\mu = 0, \forall f \in C^\infty(M)?$$

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- X_μ is a 1-cocycle in Poisson cohomology
$$\mathrm{d}_\pi X_\mu = \mathcal{L}_{X_\mu}\pi = 0.$$
- If $\nu = f\mu$ then: $X_\nu = X_\mu + \underbrace{\pi^\sharp(-\mathrm{d}\ln|f|)}_{\text{Hamiltonian vector field}}$

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- The Poisson cohomology groups are usually very hard to compute.
- The modular class is a basic invariant, with geometric meaning, which is easy to determine in many examples.

Examples

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- For a regular Poisson manifold, with transversally oriented symplectic foliation $\mathcal{F}$, one can show:

$$i^*(\mathrm{mod}\,(\mathcal{F})) = \mathrm{mod}\,(\pi), \quad i^* : H^1(\mathcal{F}) \to H^1_\pi(M).$$

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Theorem ((Evens,Lu and Weinstein,1998),(Xu,1999))

If (M,π) is unimodular then

$$H_\bullet(M,\pi) \cong H^{n-\bullet}(M,\pi).$$

Holonomy and the modular class

Theorem (Ginzburg and Golubev, 2001)

For any cotangent loop $a : I \to T^*M$:

$$\det h(a) = \exp\left(\int_a \mathrm{mod}\,(\pi)\right).$$

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Remarks

- h denotes the linear Poisson holonomy (defined using parallel transport relative to contravariant version of the Bott connection).
- Integration of Poisson vector field over cotangent paths is invariant under cotangent homotopies.

Following Evans, Lu and Weinstein (1998)

Substitute for volume form: $\eta \otimes \mu$ section of the line bundle $Q_A = \wedge^{\mathrm{top}} A \otimes \wedge^{\mathrm{top}}(T^*M)$.

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The **modular form** of the Lie algebroid A with respect to $\eta \otimes \mu$ is the A-form $\xi_A \in \Omega^1(A)$ defined by

$$\langle \xi_A, X \rangle \eta \otimes \mu = [X, \eta] \otimes \mu + \mu \otimes \mathcal{L}_{\rho(X)} \mu, \quad X \in \Gamma(A).$$

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- $$\begin{array}{rccl} \nabla : & \Gamma(A) \times \Gamma(Q_A) & \to & \Gamma(Q_A) \\ & (X, \eta \otimes \mu) & \to & [X, \eta] \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)}\mu \end{array}$$
  is a representation of the Lie algebroid A on Q_A.

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- If $\eta' \otimes \mu' = f\eta \otimes \mu$ is another section of Q_A then

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Definition

$[\xi_A] \in H^1(A)$ is the **modular class** of the Lie algebroid A.

Tangent Bundle: $A = TM$

Assume M is orientable.

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If M is orientable then $\mathrm{mod}\, TM = 0$.

Lie algebra: $A = \mathfrak{g}$

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$$\mathrm{mod}\, \mathfrak{g} = \mathrm{mod}\, (\mathfrak{g}^*, \pi_{Lie})$$

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- T^*M is a Lie algebroid over M with anchor $\rho = \pi^\sharp$ and Lie bracket

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$$\mathrm{mod}\, T^*M = 2\mathrm{mod}\,(\pi)$$

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$$\text{Pairing } H_\bullet(M) \text{ and } H_{n-\bullet}(M, \pi)$$

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- Extended to Lie-Rienart algebras by Huebschmann (1999).

Generators of Gerstenhaber algebras

Definition

An operator ∂ of degree -1 is a **generator of the Gerstenhaber algebra** $\mathfrak{X}(A)$ if, for $P \in \mathfrak{X}^p(A)$, $Q \in \mathfrak{X}(A)$,

$$[P, Q] = (-1)^p \left(\partial(P \wedge Q) - \partial P \wedge Q - (-1)^p P \wedge \partial Q\right).$$

If $\partial^2 = 0$ then $\mathfrak{X}(A)$ is **exact** or a Batalin-Vilkosky algebra.

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P. Xu (1999)

Let ∂ and ∂' be two generators of the Gerstenhaber algebra $\mathfrak{X}(A)$, such that $\partial^2 = \partial'^2 = 0$. Then

$$\partial - \partial' = i_\alpha,$$

for a closed 1-form $\alpha \in \Gamma(A^*)$.

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Let μ be a top-section of A^*. It defines another generator of $\Omega(A)$:

$$\partial_{P,\mu} = -\phi \mathrm{d}_P \phi^{-1}$$

where $\phi Q = i_Q \mu$, $Q \in \mathfrak{X}(A)$.

The **modular vector field** of a triangular Lie bialgebroid associated with μ is the vector field X_μ given by

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- If (A, P) unimodular then

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Relation between the two generalizations

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- For general triangular bialgebroids

$$\mathrm{mod}\,(A, P) \neq \frac{1}{2}\mathrm{mod}\,A^*$$

.

Y. Kosmann-Schwarzbach and A. Weinstein (2005)

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- $\mathrm{mod}^{P^\sharp}(A^*, A) = 2\,\mathrm{mod}\,(A, P)$.

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Theorem

Fix $\eta \otimes \mu$ section of $Q_A \wedge \mathrm{top}(A) \otimes \wedge^{\mathrm{top}}(T^*M)$. Then

$$\xi_{A_N} = \mathrm{d}_A(\mathrm{Tr}\, N) + N^* \xi_A.$$

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$[\mathrm{d}_A \mathrm{Tr}\, N] \in H^1(A_N)$ is the relative modular class of the Lie algebroid morphism $N : A_N \to A$.

(A, P, N): Poisson-Nijenhuis Lie algebroid

$$
\begin{array}{ccc}
(A^*, [\cdot,\cdot]_{NP}) & \xrightarrow{\quad N^* \quad} & (A^*, [\cdot,\cdot]_P) \\
\Big\downarrow P^\sharp & \searrow^{NP^\sharp} & \Big\downarrow P^\sharp \\
(A, [\cdot,\cdot]_N) & \xrightarrow{\quad N \quad} & (A, [\cdot,\cdot]_A)
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- The relative modular class of N^* has the canonical representative $\mathrm{d}_P(\mathrm{Tr}\, N^*)$, so that:

$$\mathrm{mod}^{N^*}(A^*_{N^*}, A^*) = [\mathrm{d}_P(\mathrm{Tr}\, N^*)] = [\mathrm{d}_P(\mathrm{Tr}\, N)].$$

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Definition

The **modular vector field** of the Poisson-Nijenhuis Lie algebroid (A, P, N) is

$$X_{(N,P)} = \xi_{A^*_{N^*}} - N\xi_{A^*} = \mathrm{d}_P(\mathrm{Tr}\, N) = -P^\sharp(\mathrm{d}_A \mathrm{Tr}\, N) \in \Gamma(A).$$

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- *A hierarchy of vector fields*

$$X^{i+j}_{(N,P)} = N^{i+j-1}X_{(N,P)} = \mathrm{d}_{N^iP}h_j = \mathrm{d}_{N^jP}h_i, \quad (i,j \in \mathbb{Z})$$

where $h_0 = \ln(\det N)$ *and* $h_i = \frac{1}{i}\mathrm{Tr}\, N^i, \quad (i \neq 0);$

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where $h_0 = \ln(\det N)$ *and* $h_i = \frac{1}{i} \mathrm{Tr}\, N^i, \quad (i \neq 0);$

- *A hierarchy of vector fields on M given by:*

$$X_{i+j} = -\pi_i^\sharp \mathrm{d} h_j = -\pi_j^\sharp \mathrm{d} h_i \quad (i, j \in \mathbb{Z})$$

where π_j *are Poisson structures on M.*

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The covered hierarchy of bi-Hamiltonian vector fields on M may not be generated by any Nijenhuis operator.

The two canonical examples

- Tangent bundle:
  P. Damianou and R. L. Fernandes [arXiv:math/0607784]
  Y. Kosmann and F. Magri [arXiv:math/0611202].
- Lie algebra $(\mathfrak{g}, r, N)$:
  N defines a sequence of modular forms on $\mathfrak{g}^*$, $\xi_{\mathfrak{g}^*_{N^k *}}$, which are associated with the higher brackets on $\mathfrak{g}^*$.

Relation between modular forms of $\mathfrak{g}^*$

$$\xi_{\mathfrak{g}^*_{N^k *}} = N^k \, \xi_{\mathfrak{g}^*_{N*}}.$$

Jacobi algebroid

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Schouten-Jacobi bracket on $\mathfrak{X}(A)$:

$$[P,Q]^{\phi} = [P,Q] + (p-1)P \wedge i_{\phi}Q - (-1)^{p-1}(q-1)i_{\phi}P \wedge Q,$$

$$\text{for } P \in \mathfrak{X}^{p}(A), Q \in \mathfrak{X}^{q}(A).$$

Poissonization

Consider the Lie algebroid $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$, with Lie bracket

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induced Lie algebroid structure from A by ϕ.

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$$(\hat{A}^*, [\,,\,]_{\tilde{P}}, \hat{\rho} \circ \tilde{P}^{\sharp}) \text{ is a Lie algebroid}$$

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Definition

(A, ϕ, P, N) is a **Jacobi-Nijenhuis algebroid** if $(\hat{A}, \tilde{P}, N)$ is a Poisson-Nijenhuis algebroid.

$(\hat{A}, \tilde{P}, N)$ has a modular vector field

$$\hat{X}_{(N,\tilde{P})} = \xi_{\hat{A}^*_{N^*}} - N\xi_{\hat{A}^*} = e^{-t}\mathrm{d}_P(\mathrm{Tr}\, N).$$

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$\mathrm{mod}^{(N,P)} A = \left[X_{(N,P)}\right]$ the **modular class** of (A, ϕ, P, N).

Also...

If N is non-degenerated, then

- $\hat{X}^{i+j}_{(N,\tilde{P})} = N^{i+j-1}\hat{X}_{(N,\tilde{P})} = \mathrm{d}_{N^i\tilde{P}} h_j = \mathrm{d}_{N^j\tilde{P}} h_i \in \Gamma\hat{A}$
- $X^{i+j}_{(N,P)} = N^{i+j-1}X_{(N,P)} = \mathrm{d}_{N^iP} h_j = \mathrm{d}_{N^jP} h_i \in \Gamma(A)$

$$h_0 = \ln(\det N) \quad \text{and} \quad h_i = \frac{1}{i}\mathrm{Tr}\, N^i, \quad (i \neq 0, \quad i,j \in \mathbb{Z}).$$

- These hierarchies cover two hierarchies, one on $M \times \mathbb{R}$ and another one on M.

Homework

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Homework

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