Full Screen

Close

Quit

Ergodic Solenoids

Vicente Muñoz (CSIC)

joint work with Ricardo Pérez-Marco

ArXiv: math.DG/0702501

email: vicente.munoz@imaff.cfmac.csic.es

5-8 September 2007

1. Introduction

One of the most attractive problems in differential geometry is that of realizing geometrically homology classes:

M smooth compact manifold of dimension n.
$a \in H_k(M, \mathbb{Z})$, $k < n$.
Does there exist a submanifold $S \subset M$ such that $[S] = a$?

Thom (1953) gave the answer:

- Not always. There are counterexamples (e.g. for some torsion homology classes).
- Positive answer: $\exists N \gg 0$ and $S \subset M$ such that $[S] = N\, a$.

Other relevant and related situations:

1. Existence of symplectic submanifolds in a symplectic manifold (M, ω) – Donaldson, 1994.
2. Pseudo-holomorphic curves $(S, j) \subset (M, \omega, J)$ in almost-Kähler manifolds – Gromov, 1985.
3. Embeddings of manifolds (smooth, symplectic, Riemannian, Kähler, etc.), $S \hookrightarrow M$.

Hodge Conjecture: (First version)

M compact Kähler manifold.
$a \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z})$.
Does there exist $S \subset M$ complex submanifold, s.t. $PD[S] = a$?

As stated, it is false:

- Counterexample with torsion classes (Grothendieck).
- Counterexample for M Kähler: some non-algebraic complex tori (Zucker).

Hodge Conjecture: (First version)

M compact Kähler manifold.
$a \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z})$.
Does there exist $S \subset M$ complex submanifold, s.t. $PD[S] = a$?

As stated, it is false:

- Counterexample with torsion classes (Grothendieck).
- Counterexample for M Kähler: some non-algebraic complex tori (Zucker).

Hodge Conjecture: (Current version)

(M, J, Ω) compact projective manifold (i.e. $[\Omega] \in H^2(M, \mathbb{Z})$).
$a \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z})$.
Does there exist $N \gg 0$, $S \subset M$ complex submanifold, s.t. $PD[S] = N\, a$?

Geometric representatives of non-integer homology classes:

M smooth compact manifold.

$a \in H_k(M, \mathbb{R})$ real homology class.

We look for geometric representatives of a.

An oriented submanifold $S \subset M$ represents an integer homology class $[S] \in H_k(M, \mathbb{Z})$ as follows. By duality,

$$[S] : H^k(M) \longrightarrow \mathbb{R}$$

assigns to $\omega \in \Omega^k(M)$,

$$\langle [S], \omega \rangle = \int_S \omega \ .$$

We aim for a (smooth) sub-object $S \hookrightarrow M$ defining an integration current s.t. $[S] = a$.

2. Schwartzman and Ruelle-Sullivan cycles

Schwartzman 1-dimensional cycles

Schwartzman (1957) defined, for $k = 1$, real cycles as follows.

M smooth compact manifold. A smooth parametrized curve

$$c : \mathbb{R} \longrightarrow M$$

defines by duality a Schwartzman 1-cycle

$$[c] \in H_1(M, \mathbb{R})$$

if for any 1-form $\omega \in \Omega^1(M)$, the limit

$$\langle [c], \omega \rangle = \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{1}{t - s} \int_s^t c^* \omega$$

always exists.

(This corresponds to integrating along $c([s, t])$, normalizing, and then taking the limit.)

Alternative (equivalent) definition

By compactness of M, for each pair of points $p, q \in M$ we can choose a rectifiable arc $\gamma_{p,q}$ so that for a Riemannian metric on M, the paths $(\gamma_{p,q})$ have uniformly bounded length.

The loop

$$c_{s,t} := c([s,t]) * \gamma_{c(t),c(s)}$$

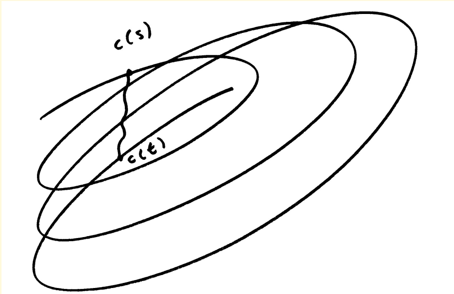
defines a homology class $[c_{s,t}] \in H_1(M, \mathbb{Z})$. Then c defines a Schwartzman 1-cycle

$$[c] \in H_1(M, \mathbb{R}) = \mathbb{R} \otimes H_1(M, \mathbb{Z})$$

if the limit

$$[c] = \lim_{\substack{t \to +\infty \\ s \to -\infty}} \frac{[c_{s,t}]}{t - s}$$

exists.



Schwartzman k-dimensional cycles

We can extend the definition to higher dimensions $k > 1$.

M smooth compact manifold.

S smooth (Riemannian) complete non-compact manifold, $x_0 \in S$.

A smooth map $f : S \to M$ defines a Schwartzman k-cycle

$$[f] \in H_k(M, \mathbb{R})$$

if for any k-form $\omega \in \Omega^k(M)$, the limit

$$\langle [f], \omega \rangle = \lim_{R \to +\infty} \frac{1}{\mathrm{Vol}_S(B_R(x_0))} \int_{B_R(x_0)} f^* \omega$$

exists, where $B_R(x_0)$ is the ball of radius R around x_0 in S.

If we can cap off the submanifolds with boundary $f(B_R(x_0))$ with a small cap C_R, $S_R = f(B_R(x_0)) \cup C_R$, then we can alternatively define

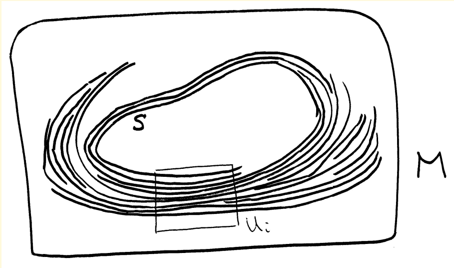
$$[f] = \lim_{R \to +\infty} \frac{[S_R]}{\mathrm{Vol}_S(B_R(x_0))} \in H_k(M, \mathbb{R}) \, .$$

This happens, for instance, when there is a trapping region, that is, a ball $B \subset M$, and a sequence $R_n \to +\infty$, s.t. $f(\partial B_{R_n}(x_0)) \subset B$.

Ruelle-Sullivan cycles

Ruelle and Sullivan (1975) introduced the following type of currents.

Let $S \subset M$ be a foliated compact subset of M, i.e. locally there are flow-boxes $U \subset M$ s.t. $U \cap S \cong D^k \times K(U)$, where $L_y = D^k \times \{y\}$ are horizontal k-disks and $K(U)$ is the transversal.



Assume that

- S has continuously oriented leaves.
- S has a transversal measure $\mu_{K(U)}$ for each transversal $K(U)$, which is invariant by holonomy.

Then there is an element

$$[S_\mu] \in H_k(M,\mathbb{R})$$

defined by duality. Fix a covering (U_i) of S, with $U_i \cap S = D^k \times K(U_i)$, (ρ_i) a partition of unity subordinated to (U_i). For $\omega \in \Omega^k(M)$, define

$$\langle [S_\mu], \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i \omega \right) \, d\mu_{K(U_i)}(y) \, .$$

Then there is an element

$$[S_\mu] \in H_k(M, \mathbb{R})$$

defined by duality. Fix a covering (U_i) of S, with $U_i \cap S = D^k \times K(U_i)$, (ρ_i) a partition of unity subordinated to (U_i). For $\omega \in \Omega^k(M)$, define

$$\langle [S_\mu], \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i \omega \right) \, d\mu_{K(U_i)}(y) \, .$$

Lemma: If S does not have compact leaves, then $[S_\mu]^2 = 0$.

Problems to solve:

(A) Can't represent all real homology classes in this way.

(B) Relationship between Schwartzman and Ruelle-Sullivan cycles.

(A) is solved by using immersed (instead of embedded) solenoids.

(B) requires solenoids whose transversal measure μ has good properties (unique ergodicity).

3. Solenoids

Definition: A k-solenoid S is a compact Hausdorff topological space with an atlas of flow-boxes (U_i) such that

$$U_i \cong D^k \times K(U_i)\,.$$

The (local) leaves are $L_y = D^k \times \{y\}$.
The (local) transversals are $K(U_i)$.

A (transversal) measure for a k-solenoid is a collection of measures $\mu = (\mu_T)$ for all transversals T such that μ_{T_1} and μ_{T_2} agree under any holonomy map $h : T_1 \to T_2$.

A measured solenoid S_μ is a solenoid endowed with a transversal measure.

A solenoid is oriented if each leaf is an oriented manifold and there is a collection of flow-boxes preserving the leaf-wise orientation.

Full Screen
Close
Quit

Homology class represented by a solenoid

M a smooth manifold.

S_μ a measured, oriented k-solenoid.

An immersion $f : S \to M$ is a globally smooth map which is an immersion for each leaf.

Then (S_μ, f) defines a real homology class

$$[S_\mu, f] \in H_k(M, \mathbb{R})$$

by duality as follows. Fix a covering (U_i) of S by flow-boxes and a partition of unity (ρ_i) subordinated to (U_i). For $\omega \in \Omega^k(M)$, let

$$\langle [S_\mu, f], \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y} \rho_i f^* \omega \right) \, d\mu_{K(U_i)}(y) \, ,$$

We call $[S_\mu, f]$ the generalized Ruelle-Sullivan class associated to the solenoid.

Full Screen
Close
Quit

Ergodic solenoids

A k-solenoid S is uniquely ergodic if:

- S admits a unique (up to positive scalar multiples) transversal measure μ.
- $\operatorname{Supp} \mu = S$.

If we endow S with a Riemannian metric (hence a k-volume along leaves) then μ can be normalized to give total volume 1 to the whole solenoid.

For a uniquely ergodic solenoid, S "contains" as much information as the measured solenoid S_μ.

If S is uniquely ergodic, then S_μ is ergodic: that is, for any transversal T and any $A \subset T$ invariant by holonomy,

$$\text{either } \mu_T(A) = 0 \text{ or } \mu_T(T - A) = 0\,.$$

If S is uniquely ergodic, then S is minimal (all leaves are dense). In particular, any transversal cuts all leaves.

Theorem: Let S be a uniquely ergodic oriented k-solenoid, and let $f : S \to M$ be an immersion. (If $k > 1$ we assume that there is a trapping region.) Then for each leaf $l \subset S$ we have that $f|_l$ is a Schwartzman k-cycle, and

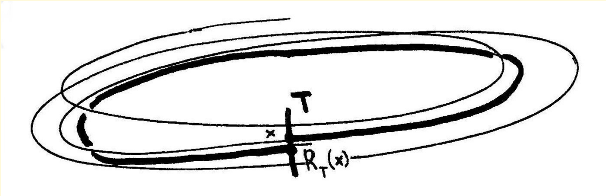
$$[f|_l] = [S_\mu, f] \in H_k(M, \mathbb{R}) .$$

Proof: This is an application of Birkhoff's ergodic theorem.

Assume for simplicity $k = 1$, so the leaf l is an (arc-length) parametrized curve $c : \mathbb{R} \to S$.

Choose a small local transversal T.

$R_T : T \to T$, Poincaré first return map.



$\varphi_T : T \to H_1(M, \mathbb{Z})$

$\varphi_T(x) = \big[f([x, R_T(x)]) * \text{short segment} \big] \in H_1(M, \mathbb{Z})$.

$l_T : T \to \mathbb{R}_+$, $l_T(x) = \text{length of } [x, R_T(x)]$.

Fix $x_0 \in T$. Let $x_i = R_T^i(x_0)$ with corresponding time t_i. Hence

$$t_{i+1} - t_i = l_T(x_i)\,.$$

Therefore

$$t_n = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = \sum_{i=0}^{n-1} l_T(R_T^i(x_0))\,.$$

By Birkhoff's ergodic theorem,

$$\lim_{n\to+\infty} \frac{1}{n} t_n = \int_T l_T(x)\, d\mu_T(x) = \mu(S) = 1\,. \tag{1}$$

Also

$$\begin{aligned}[f \circ c_{0,t_n}] &= [f \circ c_{0,t_1}] + [f \circ c_{t_1,t_2}] + \ldots + [f \circ c_{t_{n-1},t_n}] \\ &= \varphi_T(x_0) + \varphi_T(R_T(x_0)) + \ldots + \varphi_T(R_T^{n-1}(x_0))\,.\end{aligned}$$

By Birkhoff's ergodic theorem,

$$\lim_{n\to+\infty} \frac{1}{n} [f \circ c_{0,t_n}] = \int_T \varphi_T(x)\, d\mu_T(x) \in H_1(M, \mathbb{R})\,. \tag{2}$$

Now put together (1) and (2).

The Schwartzman class is(*)

$$[f|_l] = \lim_{n\to+\infty} \frac{[f\circ c_{0,t_n}]}{t_n} = \lim_{n\to+\infty} \frac{[f\circ c_{0,t_n}]/n}{t_n/n} = \int_T \varphi_T(x)\; d\mu_T(x)\,.$$

The generalized Ruelle-Sullivan class is

$$\langle [S_\mu, f], \omega\rangle = \int_T \left(\int_{[x, R_T(x)]} f^*\omega \right) d\mu_T(x) = \int_T \langle \varphi_T(x), \omega\rangle d\mu_T(x)\,,$$

for $\omega \in \Omega^1(M)$. So

$$[S_\mu, f] = \int_T \varphi_T(x)\, d\mu_T(x)\,.$$

(*) (Observe that we took $t\to\infty$. We can also take $s\to-\infty$ in a similar way.)

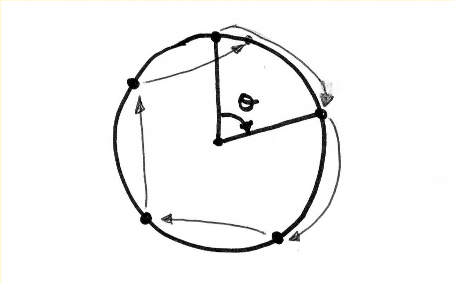
4. Realization of real homology classes

Theorem: Let M be a compact smooth manifold, $a \in H_k(M, \mathbb{R})$. Then there exists a uniquely ergodic oriented immersed solenoid (S, f) representing a. (If $k > 1$ then S has a trapping region.)

Proof: Assume $k = 1$ for simplicity.

We shall use a solenoid with a transversal $T \subset S^1$.

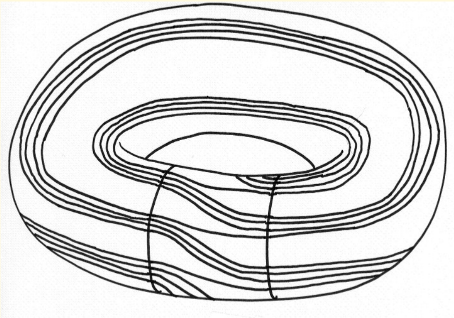
Fact: the rotation $r_\theta : S^1 \to S^1$ of irrational angle θ is uniquely ergodic.



Denjoy example: there is a map $h : S^1 \to S^1$ with the following properties:

- h is of class $C^{2-\epsilon}$.
- h leaves invariant a Cantor set $K \subset S^1$.
- h has rotation angle θ, i.e.
  $r_\theta^n(x) = x + n\,\theta$
  $h^n(x) = x + n\,\theta + o(n)$.
- h is uniquely ergodic, with a unique invariant measure μ_K supported exactly on K.

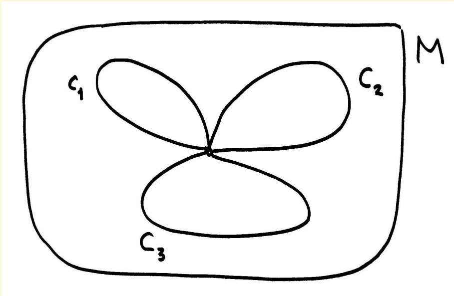
The suspension Σ_h of the map $h|_K : K \to K$ is:



Take loops $C_1, \dots, C_{b_1} \subset M$ which form a basis of $H_1(M, \mathbb{Z})$. There are real numbers $\lambda_1, \dots, \lambda_r > 0$ such that

$$a = \lambda_1 C_1 + \cdots + \lambda_r C_r$$

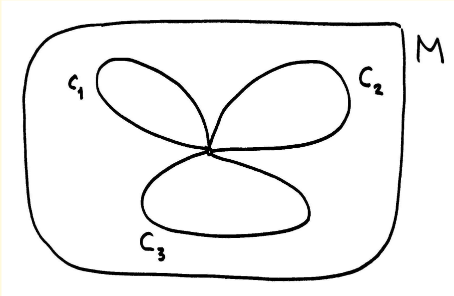
(switching orientations and reordering the cycles if necessary). By dividing by $\sum \lambda_i$, we can assume that $\sum \lambda_i = 1$.



Take loops $C_1, \dots, C_{b_1} \subset M$ which form a basis of $H_1(M, \mathbb{Z})$. There are real numbers $\lambda_1, \dots, \lambda_r > 0$ such that

$$a = \lambda_1 C_1 + \dots + \lambda_r C_r$$

(switching orientations and reordering the cycles if necessary). By dividing by $\sum \lambda_i$, we can assume that $\sum \lambda_i = 1$.



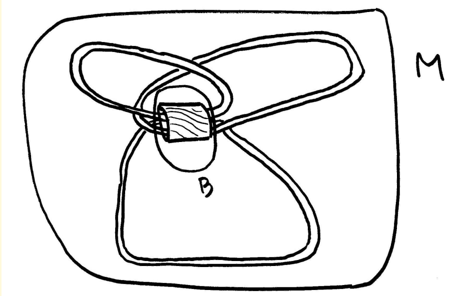
For the moment, assume $\dim M \geq 3$.

Embed $S^1 \times [0,1] \subset B$ a ball in M.

Partition the cantor set K into r disjoint compact subsets $K_1, \dots, K_r$ in cyclic order, each of which with

$$\mu_K(K_i) = \lambda_i \, .$$

Put the middle part of Σ_h inside the ball B. From the boundaries, take bands $[0,1] \times K_i$ s.t. each $[0,1] \times \{y\}$ is homotopic to C_i.



The resulting solenoid S is uniquely ergodic and, moreover, let us see that $[S_\mu, f] = a$. Let ω be any closed 1-form (we can assume that ω vanishes on B). Cover the solenoid by the central part $B \cap S$ together with the flow-boxes $[0,1] \times K_i$, $i = 1, \ldots, r$. Then

$$\langle [S_\mu, f], [\omega] \rangle = \sum_{i=1}^{r} \int_{K_i} \left(\int_{[0,1]} f^*\omega \right) d\mu_{K_i}(y) = \sum_{i=1}^{r} \int_{K_i} \langle C_i, [\omega] \rangle d\mu_{K_i}(y)$$
$$= \sum_{i=1}^{r} \langle C_i, [\omega] \rangle \mu(K_i) = \sum_{i=1}^{r} \lambda_i \langle C_i, [\omega] \rangle = \langle a, [\omega] \rangle \,,$$

proving that $[S_\mu, f] = a$.

(If $\dim M = 2$, then we cut open $S^1 \times [0,1]$ and flatten it out into a ball in M.)

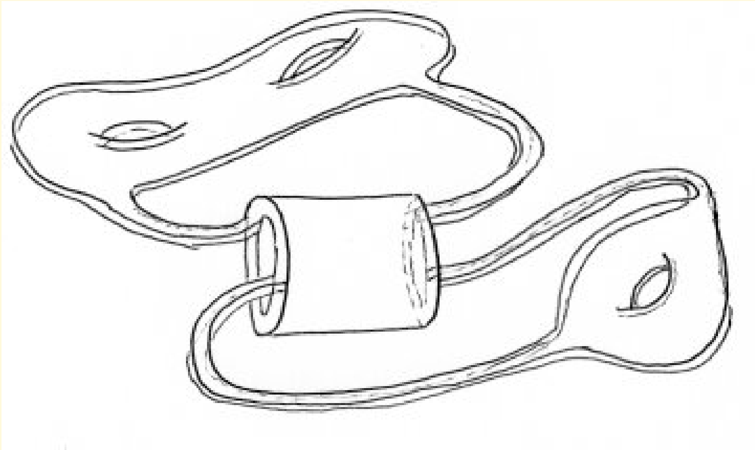
Full Screen
Close
Quit

If $k > 1$ we should use submanifolds

$$C_i \subset M$$

representing integer homology classes giving a basis of $H_k(M, \mathbb{Q}) = \mathbb{Q} \otimes H_k(M, \mathbb{Z})$, and a central part (the "mixing" locus) of the form

$$S^{k-1} \times S^1 \times [0, 1] \subset B \subset M.$$



Full Screen

Close

Quit

5. Conjecture

Solenoidal Hodge Conjecture:

Let M be a compact Kähler manifold.
$a \in H^{p,p}(M) \cap H^{2p}(M, \mathbb{R})$.
Then a is represented by a complex immersed solenoid.

Comments:

- This is a half of the Hodge Conjecture.
- It should hold disregarding the projectivity of the Kähler manifold (i.e. not requiring $[\Omega] \in H^2(M, \mathbb{Z})$).
- The conjecture holds for complex tori (using non-minimal complex solenoids).