Some connexions between analysis and geometry

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Analysis	Geometry	Algebra
Index theory	Riemann surfaces	Homological algebra
Fredholm operators	Gauss – Bonnet	Complexes
	Heat kernel	

Fredholm operators

• E, F Banach spaces.

• $A \in \mathcal{L}(E, F)$ Fredholm if A has finite dimensional kernel and cokernel.

• $A \in \mathcal{L}(E, F)$ Fredholm if and only if A is invertible modulo compact operators:

$$TA = 1 + K, \qquad AT = 1 + K'.$$

• $\mathcal{F}(E, F)$ is open in $\mathcal{L}(E, F)$.

• $\operatorname{ind} A = \dim \ker A - \dim \operatorname{coker} A \in \mathbf{Z}$ (index of A) is a homotopy invariant.

Examples

1) If E, F finite dimensional, $\mathrm{ind}\,(A) = \dim E - \dim F$.

2) If $E = F = L_2^{\geq 0}([0,1])$ (Fourier series with Fourier coefficients $a_n, n \geq 0$), $f(z)$ smooth $S_1 \to \mathbf{C}^*$, with $S_1 \simeq \mathbf{R}/\mathbf{Z}$.

$$A_f = P^{\geq 0} f.$$

Then A_f is Fredholm, and

$$\mathrm{ind}\,(A_f) = -\text{rotation number of } f.$$

The index has been expressed in topological terms.
(work out the example $f(z) = z^k \ldots$)

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Question (Gelfand): given an elliptic pseudodifferential operator A on M compact manifold, find a formula for ind A.
Answer: Atiyah-Singer index theorem.

The formula of Gauss-Bonnet

•Let S be a compact connected oriented surface.

•The fundamental topological invariant of S is its genus $g \in \mathbf{N}$= number of holes.

•$g = 0$, S is a sphere.

•$g = 1$, S is a torus. . . .

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•$\pi_1(S)$ first homotopy group generated by $a_1, b_1, \ldots a_g, b_g$ with the relation $\prod_1^g [a_i, b_i] = 1$ ($[x, y] = xyx^{-1}y^{-1}$).

•$H_1(S, \mathbf{Z})$ = abelianization of $\pi_1(S)$, $\mathbf{Z}$-module with $2g$ generators $a_1, b_1 \ldots$.

•$H^1(S, \mathbf{R})$ first real cohomology group, $\dim H^1(X, \mathbf{R}) = 2g$.

•Euler characteristic $\chi(S) = \sum_{i=1}^{2} (-1)^i \dim H^i(S, \mathbf{R}) = 2 - 2g$.

Geometric interpretation of $\chi(S)$

•$\chi(S) = S \cap S$ in $S \times S$ (self-intersection of the diagonal).

•Y vector field on S with isolated nondegenerate zeroes.

Poincaré-Hopf: $\chi(S) = \sum_{\text{zeroes of } Y} (-1)^{\text{ind}(x)}$.

(follows from the above).

•If S embedded in $\mathbf{R}^3$, the degree of the Gauss map $x \in S \to n(x) \in S_2$ is exactly $1 - g$ (direct computation, or use an affine Morse function f and its gradient field X).

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•Gauss-Bonnet: If S embedded surface in $\mathbf{R}^3$, K scalar curvature ($= 2$ for the sphere S_2!),

$$\chi(S) = \int_S \frac{K}{4\pi} dx.$$

Use the fact that $\int_S \frac{K}{2} dx = \deg(n) \operatorname{Vol}(S_2)$.
•The previous equality remains valid for any Riemannian metric on S (Gauss).

Gauss-Bonnet and Poincaré-Hopf

•Y a generic section of TS.

•Set

$$\alpha_t = \frac{1}{2\pi} \exp\left(-t\left|Y\right|^2/2\right)\left(\frac{K}{2}dx + t\omega\left(\nabla^{TS}Y, \nabla^{TS}Y\right)\right).$$

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•The α_t are closed (!) cohomologous 2-forms [3] (Mathai-Quillen).

•$\alpha_0 = \frac{K}{4\pi}dx, \alpha_{+\infty} = \sum_{\text{zeroes of } Y} (-1)^{\text{ind}(x)} \delta_x$.

•We have given another proof of Poincaré-Hopf.

The Koszul complex

•E is a two dimensional real vector space.

•E^* dual of E, generic element $\xi \in E^*$.

•Creation $\xi\wedge$, annihilation i_X act on $\Lambda^\cdot(E^*)$,

$$[\xi\wedge, i_X] = \langle \xi, X \rangle .$$

•Koszul complex

$$0 \to \Lambda^0 E^* = \mathbf{R} \xrightarrow{\xi\wedge} \Lambda^1 E^* \xrightarrow{\xi\wedge} \Lambda^2 E^* \to 0$$

is acyclic for $\xi \neq 0$ (take $X \in E, \langle \xi, X \rangle = 1$).

•If E equipped with scalar product, one can take $X = \xi^*$, and use $[\xi\wedge, i_{\xi^*}] = |\xi|^2$.

The de Rham complex

• $(\Omega^{\cdot}(S), d)$ de Rham complex

$$0 \to \Omega^0(S) \xrightarrow{d} \Omega^1(S) \xrightarrow{d} \Omega^2(S) \to 0.$$

•$d = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}, \sigma(d) = i\xi\wedge$ ($\sigma(d)$= principal symbol).

•$d^2 = 0$, $(\Omega^{\cdot}(S), d)$ elliptic complex, cohomology ($\ker d/\mathrm{Im}\ d$) $\simeq H^{\cdot}(S, \mathbf{R})$.

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•Hodge theory. g^{TS} a metric on TS. Scalar product on $\Omega^{\cdot}(S)$,

$$\langle s, s' \rangle = \int_M \langle s, s' \rangle \, dx.$$

•d^* adjoint of d.

$$\square = dd^* + d^*d = (d + d^*)^2$$

is the Laplacian: elliptic operator of order 2.

Theorem (Hodge) $\ker \square = \ker(d + d^*) \simeq H^{\cdot}(S, \mathbf{R})$.

•$d + d^*$ exchanges $\Omega^{\text{even}}(S)$ and $\Omega^{\text{odd}}(S)$.

• $(d + d^*)|_{\text{even}}$ is Fredholm, and $\operatorname{ind}(d + d^*)|_{\text{even}} = \chi(S)$ (elementary...)

•For any $t > 0$, by McKean-Singer [4],

$$\chi(S) = \mathrm{Tr_s}\left[\exp(-t\Box)\right] = \int_S \mathrm{Tr_s}\left[P_t(x,x)\right] dx.$$

($\mathrm{Tr_s}$ supertrace, use spectral theory...)

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•Weitzenböck formula:

$$\square = -\Delta + \frac{K}{4}N - \frac{K}{2}Q,$$

N number operator, $Q=1$ on 2 forms.
•As $t \to 0$, $\operatorname{Tr_s}[P_t(x,x)] \simeq \frac{1}{4\pi t}\operatorname{Tr_s}\left[e^{-tK(x)N/4+K(x)Q/2}\right]$.
•$\operatorname{Tr_s}\left[e^{-tK(x)N/4+tK(x)Q/2}\right] = 1 - 2e^{-tK/4} + e^{tK/2} \simeq tK$.
•We get $\chi(S) = \int_S \frac{K}{4\pi}dx$ (Gauss-Bonnet, cancellations in local index theory [1]).

Can one hear the shape of a surface?

•S compact surface, g^{TS} Riemannian metric on TS, Δ, the scalar Laplacian on S, is a self-adjoint elliptic operator.

•$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the spectrum of $-\Delta$.

•$p_t(x, y)$ the heat kernel associated to $e^{t\Delta}$.

•$\mathrm{Tr}\left[e^{t\Delta}\right] = \int_S p_t(x, x)\, dx$.

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•$\mathrm{Tr}\left[e^{t\Delta}\right] = \int_S p_t(x,x)\,dx$.

•As $t \to 0$,

$$p_t(x,x) = \frac{1}{4\pi t}\left(1 + \frac{tK(x)}{6} + \mathcal{O}_x\left(t^2\right)\right).$$

•As $t \to 0$,

$$\mathrm{Tr}\left[e^{t\Delta}\right] = \frac{1}{4\pi t}\left(\mathrm{Vol}(S) + \frac{t}{6}\int_S K dx\right),$$

equivalent to

$$\mathrm{Tr}\left[e^{t\Delta}\right] = \frac{\mathrm{Vol}(S)}{4\pi t} + \frac{\chi(S)}{6} + \ldots$$

Conclusion

•One can hear the volume of S.

•One can hear the genus g of S.

•Special role played by the constant terms in the asymptotic expansion.

•Note that $B_1 = \frac{1}{6}$ is the first Bernouilli number.

•Todd series $\mathrm{Td}\,(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{B_1}{2}x^2 + \dots$

Heat kernel and the loop space

- Heat semigroup $e^{(t+t')\Delta} = e^{t\Delta}e^{t'\Delta}$.
- $e^{t\Delta} = e^{t\Delta/n} \dots e^{t\Delta/n}$.
- Another expression for the trace

$$\mathrm{Tr}\left[e^{t\Delta}\right] = \int_{X^n} \underbrace{p_{t/n}(x_0, x_1) \dots p_{t/n}(x_{n-1}, x_0)} dx_0 \dots dx_{n-1}.$$

cyclic expression for the trace.

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cyclic expression for the trace.

•Compare to

$$\mathrm{Tr}\left[A^n\right] = \sum a_{i_0 i_1} a_{i_1 i_2} \ldots a_{i_{n-1} i_0}.$$

•The above sum is a sum on discrete closed loops.

•As $n \to +\infty$, the integral 'converges' to an integral on…the loop space of X.

•This measure is the Wiener measure on LX, it is invariant par rotations.

The Migdal invariant of a surface

•S a compact oriented surface of genus g.

•K a triangulation of S, with a edges.

•$A_\sigma > 0$ the area of the simplex σ.

•$\mathrm{SU}(2) \simeq S_3$ the group of special unitary transformations of $\mathbf{C}^2$, $p_t(g)$ the heat kernel on G.

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•$\mathrm{SU}(2) \simeq S_3$ the group of special unitary transformations of $\mathbf{C}^2$, $p_t(g)$ the heat kernel on G.

•To each oriented edge of K, we associate an element $g \in \mathrm{SU}(2)$.

•Each simplex σ has a holonomy $H_\sigma \in \mathrm{SU}(2)$ (ordered product of the group elements of the edges), well defined up to conjugation.

•$p_{A_\sigma}(H_\sigma) > 0$ is well-defined.

•Set

$$I = \int_{G^a} \prod p_{A_\sigma}(H_\sigma)\, dg_1 \dots dg_a.$$

Theorem. *(Migdal) The integral I only depends on the total area A on S, and does not depend on the triangulation.*

Proof. Use subdivision of the triangulation. □

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•Note that the measures on $(\mathrm{SU}(2))^a$ are compatible to each other. They can be viewed as discrete random SU (2) ‘connections’.
•Question What is I_A, what is $\lim_{A\to 0} I_A$?
Idea: Make the triangulation very small or very big.
Very small: Standard description of surface by gluing the edges of a polygon in $\mathbf{R}^2$ 2 by 2... leads to explicit computation of I_A.
Very big: The mesh goes to 0, so that each $A_\sigma \to 0$.

•For $B \in su(2), |B|$ small, as $t \to 0$,

$$p_t\left(e^B\right) \simeq \frac{\exp\left(-|B|^2/4t\right)}{(4\pi t)^{3/2}}.$$

•If each simplex has area t/n,

$$I_t \simeq C_t \int \exp\left(-n \sum |B_\sigma|^2/4t\right).$$

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•If A is a connection and F^A its curvature, the holonomy of a path bounding a domain D of small area a is $\simeq \exp\left(aF^A\right)$.

•If $B_\sigma \simeq \frac{t}{n} F^A$, then

$$n \sum |B_\sigma|^2/4t \simeq \frac{\int_S \left|F^A\right|^2}{4t}.$$

... Yang-Mills functional.

•We find that

$$I_t = \int_{\mathcal{A}} \exp\left(-\frac{\int_S \left|F^A\right|^2}{4t}\right) d\mathcal{A},$$

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•As $t \to 0$, the integral localizes on the space of flat connections.

Theorem. *(Witten [5])*

$$\lim_{t\to 0} I_t = \text{symplectic volume of moduli space of flat connections.}$$

A proof of Witten result [2]

• $f : G^{2g} \to G$ the map

$$(u_1, v_1, \ldots, u_g, v_g) \to \prod_{i=1}^{g} [u_i, v_i] .$$

• G acts on G^{2g} and G by conjugation.

• f is a G equivariant map

$$f(g.x) = g.f(x) .$$

Question: What is the image of the measure $dg_1 \ldots dg_{2g}$ by f?

Answer: Change of variable formula.

(1) $$f\left(g.x\right) = g.f\left(x\right).$$

•Differentiate (1) in the variable g at $g = 1$, when $f\left(x\right) = 1$. If $A \in su\left(2\right)$,

(2) $$\left\langle df\left(x\right), A.x\right\rangle = 0.$$

•We have the finite dimensional complex,

(3) $$0 \to su\left(2\right) \xrightarrow{\partial} su\left(2\right)^{2g} \xrightarrow{\partial} su\left(2\right) \to 0.$$

•The Euler characteristic of this complex is $3\left(2 - 2g\right)\ldots$

Explanation

•The set $\left\{x \in G^{2g}, f(x) = 1\right\}$ is the set of representations of $\pi_1(S) \to G$.

•The complex

$$0 \to su(2) \xrightarrow{\partial} su(2)^{2g} \xrightarrow{\partial} su(2) \to 0$$

is a combinatorial complex whose cohomology is the cohomology of the flat adjoint bundle on S.

•One can now use the standard change of variables formula....

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