THE TALE OF A GEOMETRIC INEQUALITY

S.Tabachnikov
(Pennsylvania State University)

I want to tell you the story of a rather mysterious geometric inequality. The story begins a few years ago, when a colleague who loves mathematical puzzles asked me this one:

How many moves does a rook need to visit every square of a chess board?

Instead of solving this rather easy problem, I did what mathematicians often do when in doubt: generalized. I argued as follows.

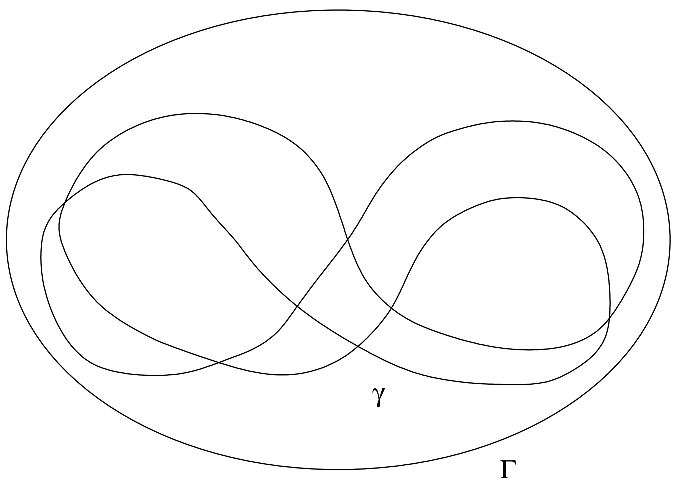

FIGURE 0.1

The path of the rook is a curve γ (possibly, self-intersecting) inside an 8×8-square. The length of γ should be at least 64 since all squares are visited. Next, I assumed that the rook turned each time by 90 degrees (and never 180). Then the total turn of γ is $n\pi/2$, where n is the number of moves. Thus I traded the original problem for the following one:

What is the smallest total turn of a curve of given length inside a given convex domain in the plane?

Consider a smooth closed convex plane curve Γ and a smooth, possibly self-intersecting (the technical term is "immersed") closed curve γ inside it, see Fig.0.1.

MASS colloquium lecture, November 1999.

For obvious reasons, I will call Γ the *cell* and γ the *DNA*.

Define the *total curvature* of a closed curve as the integral of the *absolute value* of the curvature with respect to the arc length parameter along the whole curve. The total curvature is what I called "total turn" above (unlike the integral of the curvature, which can be positive or negative, the total curvature is not necessarily a multiple of 2π). The *average curvature* of a curve is the total curvature divided by its length.

My reflections on the rook on the chess board problem lead me to the following statement, which I call the *DNA geometric inequality*.

THEOREM 0.1. (i) *The average curvature of the cell is not greater than that of the DNA.* (ii) *The equality holds if and only if the DNA coincides with the cell, possibly traversed more than once.*

Although I call this a theorem, it was only a conjecture: hard as I tried, I could neither prove nor disprove the DNA inequality. Then it occurred to me that I had come across a similar result a long time ago. In 1973, the year I graduated from high school, I took part in the Moscow Mathematical Olympiad, where the following problem was given:

A lion runs inside the round arena of a circus. The radius of the circle is 10 meters. Moving along a polygonal line, the lion covers the total distance of 30 kilometers. Prove that the total angle of turn of the lion is not less than 2998 radians.

This is a particular case of the DNA inequality: the one in which the curve Γ (the arena) is a circle (albeit the fact that the curve γ here is not closed). In fact, for Γ a circle, the DNA inequality was proved by I. Fáry [1] about 50 years ago, see [**4**].

Anxious to prove Theorem 0.1 in full generality and unable to do it myself, I started to ask around. Finally, in 1994, J. Lagarias and T. Richardson found a proof of statement (i) (see [**7**]). This proof is quite involved, and one cannot help but hope that the "proof from the Book" will be shorter and more transparent. Another problem with the Lagarias–Richardson approach is that it does not prove statement (ii) of Theorem 0.1, which remains a conjecture.

My goal here is to give four proofs of the Fáry theorem, that is, Theorem 0.1 with Γ a circle; these proofs make use of very different ideas. I will also discuss multidimensional generalizations of the DNA inequality.

Without loss of generality, assume that Γ is the unit circle. Denote the total curvature by c and the length by l. Since $c(\Gamma) = 2\pi = l(\Gamma)$, the inequality to establish reads:

$$c(\gamma) \geq l(\gamma),$$

[1] Another result of his, the Fáry-Milnor theorem, is better known: the total curvature of a nontrivial knot in 3-space is not less than 4π, see [**5, 8**].

the equality holding if and only if γ is a multiple of the unit circle, i.e., the unit circle traversed once or several times.

Sometimes it is convenient to consider γ as a smooth curve, and sometimes, as a polygonal line (the total curvature c of the latter is the sum of its external angles); I will freely go from one set-up to the other.

1. Proof by Rolling

Assume that γ is a polygonal line with sides e_i of length l_i. Starting with $i = 1$, rotate the side e_{i+1} about its common end-point v_i with e_i so that e_{i+1} becomes the extension of e_i. The rotation angle is equal to the exterior angle α_i of the polygonal line γ at the vertex v_i, see Fig.1.1. In this way one straightens the polygonal line into a segment. In other words, one rolls all of γ along a straight line.

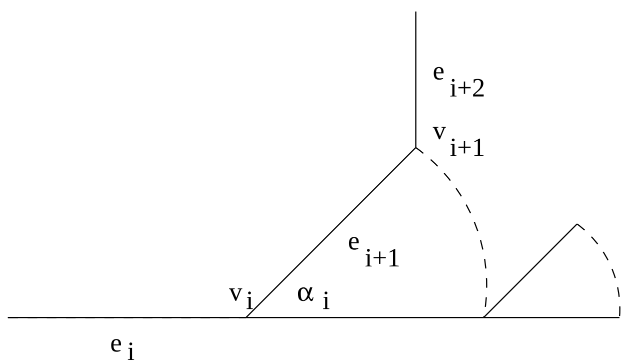

FIGURE 1.1

Let the plane containing γ roll along with γ. The total displacement of the center O of the unit circle Γ is a horizontal segment of length $\sum l_i = l(\gamma)$. The trajectory of the point O consists of arcs of circles of radii not greater than 1 (supporting the angles α_i); the length of such an arc does not exceed α_i. Clearly the length of the trajectory of O is not less than its total displacement, that is,

$$c(\gamma) = \sum \alpha_i \geq l(\gamma),$$

as claimed.

This proof is the "official solution" of the lion on the circus arena problem from the Moscow Mathematical Olympiad as given in [**6**].

2. Proof by the Triangle Inequality

We continue to assume that γ is a polygonal line; as before, α_i is the exterior angle at vertex v_i and l_i is the length of the side $e_i = v_{i-1}v_i$. One wants to show that $\sum \alpha_i \geq \sum l_i$.

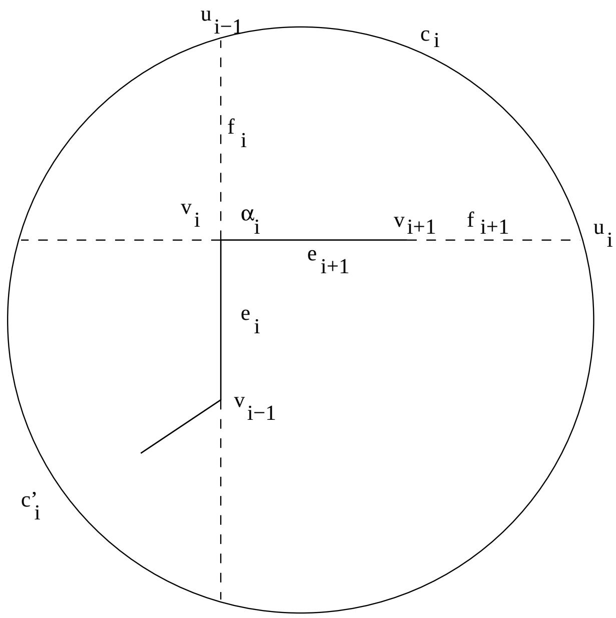

FIGURE 2.1

Denote the intersection point of the (oriented) side e_i with the circle Γ by u_{i-1}, and let $f_i = |v_i u_{i-1}|$, see Fig.2.1. Denote the length of the arc $u_{i-1}u_i$ by c_i. Then, by the triangle inequality for the "triangle" $u_{i-1}v_iu_i$, we have

$$f_{i+1} + l_{i+1} \leq f_i + c_i.$$

If we sum over i, the f-terms will cancel, and we obtain:

$$\sum l_i \leq \sum c_i.$$

Now change the orientation of γ to the opposite one, and let c_i' be the length of the corresponding arc of the circle Γ. As before,

$$\sum l_i \leq \sum c_i'.$$

By elementary geometry of the circle, $c_i + c_i' = 2\alpha_i$. Adding the two previous inequalities, we get:

$$\sum l_i \le \sum \alpha_i,$$

as required.

The same argument involving the triangle inequality proves Theorem 0.1 in the case when γ does not have inflection points, that is, has a nonvanishing curvature.

3. Proof by Calculus

In this proof, $\gamma(t)$ is a smooth curve parameterized by arc length. Choose the origin at the center of the cell Γ. Then $|\gamma'(t)| = 1$, and $|\gamma''(t)| = |k(t)|$ is the absolute curvature of γ. In addition, $|\gamma(t)| \le 1$ for all t. We have:

$$l = \int_0^l \gamma' \cdot \gamma' \; dt = -\int_0^l \gamma \cdot \gamma'' \; dt \le \int_0^l |\gamma| \; |\gamma''| \; dt \le \int_0^l |k(t)| \; dt,$$

where $\cdot$ is the scalar product of vectors; the second equality is integration by parts). This is the desired result.

This proof is due to G. Chakerian, see [**3**].

4. Proof by Integral Geometry

An oriented line r in the plane is uniquely determined by its direction α and its signed distance p from the origin (which is again chosen to be the center of the circle Γ); we write $r(p, \alpha)$, see Fig.4.1.

The set of lines that intersect Γ is described by the inequalities

$$0 \le \alpha \le 2\pi, \quad -1 \le p \le 1.$$

Consider the measure $dp \; d\alpha$ on the set of lines (this is, up to a factor, the unique motion-invariant measure on this set). Given an oriented closed immersed curve γ, define a locally constant function $n(p, \alpha)$ on the set of lines as the number of intersection points of γ with the line $r(p, \alpha)$. The Crofton formula of integral geometry reads:

$$l(\gamma) = \frac{1}{4} \int \int n(p, \alpha) \; dp \; d\alpha,$$

see [**9**].

Define another locally constant function $k(\alpha)$ to be the number of oriented tangent lines to γ in the direction α. Then one has the following integral formula for the total curvature:

$$c(\gamma) = \int_0^{2\pi} k(\alpha) \; d\alpha.$$

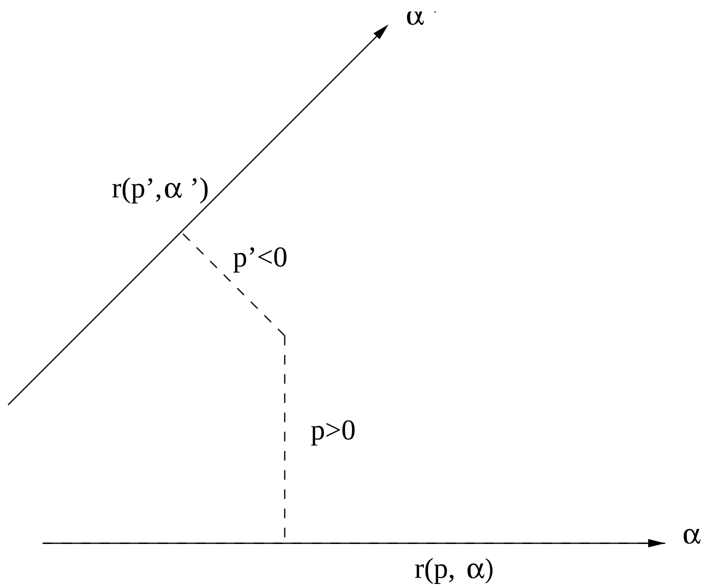

FIGURE 4.1

The crucial observation is that

$$n(p,\alpha) \leq k(\alpha) + k(\alpha+\pi)$$

for all p, α. Indeed, between two consecutive intersections of γ with the line $r(p,\alpha)$, the tangent line to γ has the direction α or $\alpha+\pi$ at least once (essentially, this is Rolle's theorem).

It remains to integrate this inequality, keeping in mind that we have $n(p,\alpha)=0$ for $|p|>1$:

$$\begin{aligned} l(\gamma) &= \frac{1}{4}\int_0^{2\pi}\int_{-1}^{1} n(p,\alpha)\ dp\ d\alpha \leq \frac{1}{2}\int_0^{2\pi} (k(\alpha)+k(\alpha+\pi))\ d\alpha \\ &= \int_0^{2\pi} k(\alpha)\ d\alpha = c(\gamma), \end{aligned}$$

as claimed.

This argument, essentially due to Fáry, can be found in [**9**]. The integral geometry approach is the most conceptual of all four; it also yields the DNA inequality in the case when the cell Γ is a curve of constant width.

5. Generalizations

One may change the Euclidean metric to a Riemannian one and ask whether the DNA inequality still holds. The first case to investigate is that of the metric of constant curvature, that is, the spherical or hyperbolic one. To the best of my knowledge, this problem is open, even the constant curvature version of the Fáry theorem is not available in the literature.

Another generalization concerns the dimensions involved: one may consider a convex n-dimensional "cell" Γ and an immersed m-dimensional "DNA" γ (the "physical" case being $n = 3, m = 1$). Note that all the above four proofs of the Fáry inequality extend to the case when γ is a curve inside the n-dimensional unit ball (curiously, what Fáry proved in the multidimensional case [**4**] was a weaker inequality $4c(\gamma) \geq \pi l(\gamma)$; the multidimensional inequality $c(\gamma) \geq l(\gamma)$ appeared in [**2**]).

A generalization to two-dimensional DNA is contained in [**4**]: if Γ is the unit sphere in 3-space and γ a closed immersed surface, then $\pi A(\gamma) \leq 4K(\gamma)$, where A is the area and K the total (absolute) Gauss curvature. Still another generalization is found in [**1**]: if Γ is the unit sphere in n-space and γ an closed immersed m-dimensional submanifold, then

$$mV(\gamma) \leq \int_\gamma |H| \, dv,$$

where V is the m-dimensional volume and H is the mean curvature vector (the proof is not much different from the one by calculus above).

All these multidimensional results go along Fáry's lines: the outer surface is the sphere. What about a more general case, say, a closed curve inside an ovaloid? It is not even clear what the hypothetical lower bound for the average curvature of such a curve should be...

And, before I finish, what is the answer to the rook on the chess board puzzle?

References

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