# QUANTIZATION OF NON-GEOMETRIC FLUX BACKGROUNDS

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## String geometry

- Strings see geometry in different ways than particles do:

  T-duality, non-geometric backgrounds, ...

- Probes of Planck scale quantum geometry:

  Spacetime uncertainty $\Delta x \geq \ell_P$ related to noncommutative spacetime structure?

- Strings and noncommutative geometry:
  - D-branes in $B$-fields provide realisations of noncommutative spaces
  - Low-energy dynamics described by noncommutative gauge theory, Seiberg–Witten maps, ...

    (Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; ...)

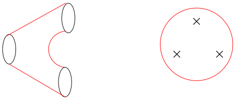
# Noncommutative string geometry

- Open string noncommutative gauge theory:
  - 2-point function on boundary of disk: ordering

  - 2-tensor ($B$-field) deforms to noncommutative 2-bracket
  - Quantization produces star-products of fields

- Closed string nonassociative gravity

  (Lüst '10; Blumenhagen & Plauschinn '11; Blumenhagen *et al.* '11)



  - 3-point function on sphere: orientation
  - 3-tensor ($R$-flux) deforms to nonassociative 3-bracket
  - Quantization (Dynamical star-products, categorified Weyl quantization, quasi-Hopf cochain quantization)
  - Seiberg–Witten maps

## Nonassociativity and gravity in string theory

- Nonassociative gauge theory:

  *Open string* nonassociative spaces with 3-form $H = \mathrm{d}B \neq 0$

  (Cornalba & Schiappa '02; Ho '01; Herbst, Kling & Kreuzer '01)

  Described locally by associative algebras and noncommutative gerbes (Aschieri *et al.* '10)

- Noncommutative gravity:

  Differential geometry/general relativity on noncommutative spacetime (Aschieri *et al.* '05)

  Twisted diffeomorphisms do not arise as physical symmetries of string theory (Álvarez-Gaumé, Meyer & Vázquez-Mozo '06)

## Non-geometric flux compactification

$T^3$ with $H$-flux gives geometric and non-geometric fluxes via T-duality

(Hull '05; Shelton, Taylor & Wecht '05)

$$H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc}$$

- $H$-flux: $H = \mathrm{d}B$ , gerbes
- Metric flux: $\mathrm{d}e^a = -\frac{1}{2} f^a{}_{bc}\, e^b \wedge e^c$ , twisted torus
- $Q$-flux: T-folds: stringy transition functions
  Fibration $T^3 \xrightarrow{T^2_\theta} S^1$ of noncommutative 2-tori
  $[x^i, x^j] = Q^{ij}{}_k\, x^k$ (Mathai & Rosenberg '04; Grange & Schäfer-Nameki '07)
- $R$-flux: Not even locally geometric
  (Topological) nonassociative 3-tori $[x^i, x^j, x^k] = R^{ijk}$
  (Bouwknegt, Hannabuss & Mathai '06; Ellwood & Hashimoto '06)

## Geometrisation of non-geometry

- Generalised geometry: Non-geometric fluxes through T-duality transforms of sections of $C = TM \oplus T^*M$

  (Grange & Schäfer-Nameki '07; Graña *et al.* '08; Halmagyi '09)

- Doubled geometry/field theory: $x^i \longrightarrow (x^i, \tilde{x}_i)$ , $\partial_i \longrightarrow (\partial_i, \tilde{\partial}^i)$; implement projection $\tilde{\partial}^i = 0$ (Andriot *et al.* '12)

- Membrane $\sigma$-model: Open strings do not decouple from gravity in $R$-space (Ellwood & Hashimoto '06)

  Target space doubling $M \longrightarrow T^*M$, geometric 3-form $R$-flux

  Membrane $\Longrightarrow$ closed strings with non-geometric flux
  $+$ open string $R$-twisted Poisson $\sigma$-model whose perturbative quantization gives nonassociative dynamical star-product of fields, Jacobiator quantizes 3-bracket (Mylonas, Schupp & RS '12)

## Poisson $\sigma$-model

- AKSZ $\sigma$-models: Action functionals in BV formalism for $\sigma$-models with target space a symplectic Lie $n$-algebroid $E \longrightarrow M$
  (Alexandrov *et al.* '95)

- Poisson $\sigma$-model: $n = 1$ , $E = T^*M$
  Most general 2D TFT from AKSZ construction

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big( \xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j \Big)$$

  $X : \Sigma_2 \longrightarrow M,\ \ \xi \in \Omega^1(\Sigma_2, X^*T^*M),\ \ \Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$

- Perturbative expansion $\Longrightarrow$ Kontsevich formality maps
  (Kontsevich '03; Cattaneo & Felder '00)

  On-shell: $[\Theta, \Theta]_S = 0$ Poisson structure

## Courant $\sigma$-model

- $n = 2$ , $E =$ Courant algebroid with fibre metric $h_{IJ} = \langle \psi_I, \psi_J \rangle$ , anchor matrix $\rho(\psi_I) = P_I{}^i(x)\,\partial_i$ , 3-form $T_{IJK}(x) = [\psi_I, \psi_J, \psi_K]_E$

- Correspond to TFT on 3D membrane worldvolume $\Sigma_3$ (Hofman & Park '02; Ikeda '03; Roytenberg '07):

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big( \phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, h_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - P_I{}^i(X)\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big)$$

$$X : \Sigma_3 \longrightarrow M, \;\; \alpha \in \Omega^1(\Sigma_3, X^*E), \;\; \phi \in \Omega^2(\Sigma_3, X^*T^*M)$$

- Standard Courant algebroid: $E = C = TM \oplus T^*M$ with natural frame $(\psi_I) = (\partial_i, \mathrm{d}x^i)$ , dual pairing $\langle \partial_i, \mathrm{d}x^j \rangle = \delta_i{}^j$

## $\sigma$-model for geometric fluxes

- $C$ twisted by $H$-flux $H = \frac{1}{6}\, H_{ijk}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ with $H$-twisted Courant–Dorfman bracket:

$$\big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_H \;:=\; \big([Y_1,Y_2]_{TM}\,,\,\mathcal{L}_{Y_1}\alpha_2-\mathcal{L}_{Y_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1)-\alpha_1(Y_2)\big)+H(Y_1,Y_2,-)\big)$$

and projection $\rho\,:\, C \longrightarrow TM$

- Brackets: $[\partial_i,\partial_j]_H = H_{ijk}\,\mathrm{d}x^k$ , $[\partial_i,\partial_j,\partial_k]_H = H_{ijk}$

- Topological membrane with $(\alpha^I) = (\alpha^1,\dots,\alpha^d,\xi_1,\dots,\xi_d)$ (Park '01):

$$S^{(2)}_{\rm AKSZ} = \int_{\Sigma_3}\Big(\phi_i\wedge\mathrm{d}X^i+\alpha^i\wedge\mathrm{d}\xi_i-\phi_i\wedge\alpha^i+\frac{1}{6}\,H_{ijk}(X)\,\alpha^i\wedge\alpha^j\wedge\alpha^k\Big)$$

## $H$-twisted Poisson $\sigma$-model

- When $\Sigma_2 := \partial\Sigma_3 \neq \varnothing$, add boundary term $\int_{\Sigma_2} \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j$ to get boundary/bulk open topological membrane (Hofman & Park '02)

- Integrate out $\phi_i$:

$$\widetilde{S}^{(1)}_{\rm AKSZ} = \int_{\Sigma_2} \Big(\xi_i \wedge {\rm d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) + \int_{\Sigma_3} \frac{1}{6}\,H_{ijk}(X)\,{\rm d}X^i \wedge {\rm d}X^j \wedge {\rm d}X^k$$

- On-shell: $[\Theta,\Theta]_S = \bigwedge^3 \Theta^\sharp(H)$ $H$-twisted Poisson structure (Jacobi identity for bracket violated)

$$H \longmapsto Q \longmapsto R$$

- Closed strings on $Q$-space $M = T^3 \xrightarrow{T^2} S^1$ (locally), worldsheet $\mathcal{C} = \mathbb{R} \times S^1 \ni (\sigma^0, \sigma^1)$, winding number $\tilde{p}^{\,3}$, twisted boundary conditions at $\sigma'^{\,1}$

- Closed string noncommutativity (Lüst '10; Blumenhagen *et al.* '11; Condeescu, Florakis & Lüst '12; Andriot *et al.* '12):

$$\{x^i, x^j\}_Q = Q^{ij}{}_k \, \tilde{p}^{\,k} \ , \qquad \{x^i, \tilde{p}^{\,j}\}_Q = 0 = \{\tilde{p}^{\,i}, \tilde{p}^{\,j}\}_Q$$

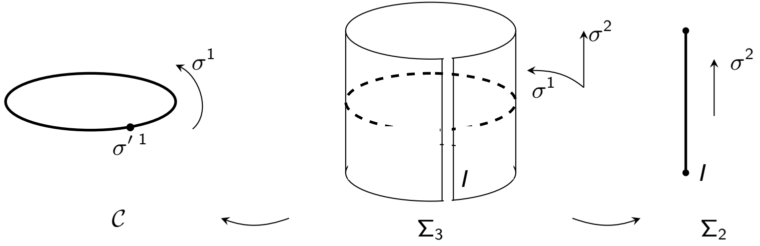
- T-duality sends $Q^{ij}{}_k \longmapsto R^{ijk}$ , $\tilde{p}^{\,k} \longmapsto p_k$:

$$\{x^i, x^j\}_\Theta = R^{ijk} \, p_k \ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \ , \qquad \{p_i, p_j\}_\Theta = 0$$

- Twisted Poisson structure on $T^*M$: Closed string nonassociativity $\{x^i, x^j, x^k\}_\Theta = R^{ijk}$; transgresses to Poisson structure on loop space of $M$ (Sämann & RS '12)

## Closed/open string duality in $Q$-space

- CFT correlators: Insert twist field at $\sigma'^{\,1} \in S^1$ creating branch cut
- Worldsheet $\mathcal{C} = \mathbb{R} \times S^1 \implies$ Worldvolume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$
- "Branch surface" $\equiv$ open string worldsheet $\Sigma_2$:



## $\sigma$-model for non-geometric fluxes

- $C$ twisted by trivector $R = \frac{1}{6}\, R^{ijk}(x)\, \partial_i \wedge \partial_j \wedge \partial_k$, with Roytenberg bracket:

$$\big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_R \;:=\; \Big([Y_1,\,Y_2]_{TM} + R(\alpha_1,\alpha_2,-)\,,\; \mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1) - \alpha_1(Y_2)\big)\Big)$$

- Brackets: $[\mathrm{d}x^i, \mathrm{d}x^j]_R \;=\; R^{ijk}\,\partial_k \;\;,\;\; [\mathrm{d}x^i, \mathrm{d}x^j, \mathrm{d}x^k]_R \;=\; R^{ijk}$

- Integrate out $\phi_i$:

$$S_R^{(2)} \;=\; \int_{\Sigma_2} \xi_i \wedge \mathrm{d}X^i + \int_{\Sigma_3} \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}(X)\, \xi_i \wedge *\xi_j$$

- For constant $R^{ijk}$ and $g^{ij}$, e.o.m. for $X^i \Longrightarrow \xi_i \;=\; \mathrm{d}P_i$

# Generalized Poisson $\sigma$-model

- Pure boundary theory, linearize with auxiliary fields $\eta_I$:

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\,\Theta^{IJ}(X)\,\eta_I \wedge \eta_J\Big) + \int_{\Sigma_2} \frac{1}{2}\,G^{IJ}\,\eta_I \wedge *\eta_J$$

$$X = (X^1,\dots,X^d,P_1,\dots,P_d) \in T^*M \quad , \; \Theta = \begin{pmatrix} R^{ijk}\,p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \; ,$$

$$G^{IJ} = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

Effective target space = phase space

- $H$-twisted Poisson bivector: $\Pi := [\Theta,\Theta]_S = \bigwedge^3 \Theta^\sharp(H)$ ,
  with $H = \mathrm{d}B$, $B = \frac{1}{6}\,R^{ijk}\,p_k\,\mathrm{d}p_i \wedge \mathrm{d}p_j$

- Noncommutative/nonassociative phase space:

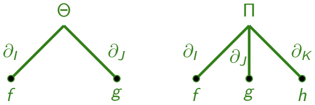
$$\{x^I,x^J\}_\Theta = \Theta^{IJ}(x)\,,\; \{x^I,x^J,x^K\}_\Theta := \Pi(x^I,x^J,x^K) = \begin{pmatrix} R^{ijk} & 0 \\ 0 & 0 \end{pmatrix}$$

## Path integral quantization

- Suitable functional integrals reproduce Kontsevich's graphical expansion for global deformation quantization (Cattaneo & Felder '00). Formality maps $U_n$ : multivector fields $\longrightarrow$ differential operators:

$$U_n(\mathcal{X}_1, \dots, \mathcal{X}_n) \;=\; \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \dots, \mathcal{X}_n)$$

- Star-product and 3-bracket:




$$f \star g \;=\; \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} \, U_n(\Theta, \dots, \Theta)(f, g) \;=:\; \Phi(\Theta)(f, g)$$

$$[f, g, h]_\star \;=\; \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} \, U_{n+1}(\Pi, \Theta, \dots, \Theta)(f, g, h) \;=:\; \Phi(\Pi)(f, g, h)$$

## Formality conditions

- $U_n$ define $L_\infty$-morphisms relating Schouten brackets to Gerstenhaber brackets (Kontsevich '03)

- $[\Phi(\Theta),\star]_G \ = \ \mathrm{i}\,\hbar\,\Phi([\Theta,\Theta]_S)$ quantifies nonassociativity:

$$(f\star g)\star h - f\star(g\star h) \ = \ \tfrac{\hbar}{2\,\mathrm{i}}\,\Phi(\Pi)(f,g,h) \ = \ \tfrac{\hbar}{2\,\mathrm{i}}\,[f,g,h]_\star$$

- $[\Phi(\Pi),\star]_G \ = \ \mathrm{i}\,\hbar\,\Phi([\Pi,\Theta]_S)$ encodes quantum Leibniz rule for Nambu–Poisson structure $\{f,g,h\}_\Pi \ := \ \Pi(\mathrm{d}f,\mathrm{d}g,\mathrm{d}h)$:

$$[f\star g,h,k]_\star-[f,g\star h,k]_\star+[f,g,h\star k]_\star \ = \ f\star[g,h,k]_\star+[f,g,h]_\star\star k$$

- Higher derivation properties encode quantum fundamental identity, giving quantized Nambu–Poisson structure

## Dynamical nonassociativity

- Explicit dynamical nonassociative star-product:

$$f \star g \ = \ f \star_p g \ := \ \cdot\Big( \mathrm{e}^{\frac{\mathrm{i}\,\hbar}{2}\, R^{ijk}\, p_k\, \partial_i \otimes \partial_j}\ \mathrm{e}^{\frac{\mathrm{i}\,\hbar}{2}\, \big(\partial_i \otimes \tilde{\partial}^i - \tilde{\partial}^i \otimes \partial_i\big)} (f \otimes g) \Big)$$

- Replace dynamical variable $p$ with constant $\tilde{p}$
  $\Longrightarrow$ associative Moyal-type star-product $\tilde{\star} \ := \ \star_{\tilde{p}}$

- 3-bracket:

$$[f,g,h]_\star \ = \ \frac{4\,\mathrm{i}}{\hbar} \Big[ \tilde{\star} \Big( \sinh\big(\tfrac{\hbar^2}{4}\, R^{ijk}\, \partial_i \otimes \partial_j \otimes \partial_k\big)(f \otimes g \otimes h) \Big) \Big]_{\tilde{p} \to p}$$

- Trace property: $\int [f,g,h]_\star \ = \ 0$

- Agrees with 3-product from closed string vertex operators
  (Blumenhagen *et al.* '11)

## Seiberg–Witten maps and noncommutative gerbes

- Poisson $\Theta$, quantum Moser lemma $\equiv$ covariantizing map $\mathcal{D}$ (Jurčo, Schupp & Wess '00):

$$
\begin{array}{ccccc}
B\ : & & \Theta & \xrightarrow{\text{quant}} & \star \\
\downarrow \rho & & \downarrow \rho & & \downarrow \mathcal{D} \\
B+F\ : & & \Theta' & \xrightarrow[\text{quant}]{} & \star'
\end{array}
$$

$$\Theta' \ = \ \Theta\,(1+\hbar\,F\,\Theta)^{-1} \ , \ \ \rho \ = \ \text{flow generated by } \ \Theta(A,-) \ ,$$
$$\mathcal{D}(f\star g) \ = \ \mathcal{D}f\star'\mathcal{D}g \ , \ \ \mathcal{D}x \ =: \ x+\hat{A}$$

- $H$-twisted $\Theta$: $H \ = \ \mathrm{d}B_\alpha \ , \ \ B_\beta - B_\alpha \ = \ \mathrm{d}A_{\alpha\beta}$
  Untwisted $\Theta_\alpha \ := \ \Theta\,(1-\hbar\,B_\alpha\,\Theta)^{-1}$

  $\Theta \ \xrightarrow{\text{quant}}$ nonassociative $\star$
  $\Theta_\alpha, \Theta_\beta \ \xrightarrow{\text{quant}}$ associative $\star_\alpha, \star_\beta$ related by $\mathcal{D}_{\alpha\beta}$ (Aschieri *et al.* '10)

## Generalised Seiberg–Witten maps for non-geometric fluxes

- Trivial gerbe on $T^*M$: $\alpha = $ constant momentum $\tilde{p}$:

$$\Theta_{\tilde{p}} = \begin{pmatrix} \hbar\, R^{ijk}\,\tilde{p}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \qquad , \qquad B_{\tilde{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk}\,(p_k - \tilde{p}_k) \end{pmatrix}$$

  $\mathcal{D}_{\tilde{p}\tilde{p}'} : \tilde{\star} \longrightarrow \tilde{\star}'$ generated by $A_{\tilde{p}\tilde{p}'} = R^{ijk}\, p_i\,(\tilde{p}_k - \tilde{p}'_k)\,\mathrm{d}p_j$;
  for $\tilde{p} = 0$ canonical Moyal star-product $\star_0$

- $\mathcal{D}_{\tilde{p}0}$ : associative $\longrightarrow$ nonassociative can be computed explicitly:

$$f \star g = \big[\mathcal{D}_{\tilde{p}} f \star_0 \mathcal{D}_{\tilde{p}} g\big]_{\tilde{p}\to p}$$

- Nonassociative generalization of Seiberg–Witten map:
  $A = \tilde{a}^i(x,p)\,\mathrm{d}p_i$:
  - Quantized diffeomorphisms: $\Theta(A,-) = \tilde{a}^i(x,p)\,\partial_i$
    "nonassociative gravity"
  - Quantized Nambu–Poisson maps: $A = R(a_2,-)$ , $a_2 \in \Omega^2(M)$
    higher gauge theory

## Categorified Weyl quantization

- Convolution quantization of Lie 2-algebras

- Reduction of standard $R$-space Courant algebroid:

$$[x^i, x^j]_Q = \mathrm{i}\, R^{ijk}\, \tilde{p}_k \ , \ [x^i, \tilde{p}_j]_Q = 0 = [\tilde{p}_i, \tilde{p}_j]_Q \ , \ \langle x^i, p_j \rangle = \delta^i{}_j$$

- Induces skeletal 2-term $L_\infty$-algebra

$$V = \big( V_1 = \mathbb{R} \xrightarrow{\ \mathsf{d}\ } V_0 = \mathfrak{g} \big)$$

with classifying 3-cocycle $j : \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathbb{R}$ ,
$j(x^i, x^j, x^k) = R^{ijk}$

- Integrates to Lie 2-group

$$\mathscr{G}_1 = G \times U(1) \ \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \ \mathscr{G}_0 = G$$

with associator integrating $j$

## Quasi-Hopf cochain quantization

- Nonabelian Lie algebra $\mathfrak{g}$ of Bopp shifts generated by vector fields:

$$P_i \ = \ \partial_i \ , \ \tilde{P}^i \ = \ \tilde{\partial}^i \ , \ M_{ij} \ = \ p_i\,\partial_j - p_j\,\partial_i$$

- Quasi-Hopf deformation of $U(\mathfrak{g})$ by cochain twist:

$$\mathcal{F} \ = \ \exp\Big[-\tfrac{\mathrm{i}\,\hbar}{2}\,\Big(\tfrac{1}{2}\,R^{ijk}\,M_{jk}\otimes P_i + P_i\otimes\tilde{P}^i - \tilde{P}^i\otimes P_i\Big)\Big]$$

- Quantization functor on category of quasi-Hopf module algebras generates nonassociative algebras through associator

$$\phi \ = \ \partial\mathcal{F} \ = \ \exp\big(\tfrac{\hbar^2}{2}\,R^{ijk}\,P_i\otimes P_j\otimes P_k\big)$$